COURSE NAME – EM FIELD

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Basics of Gradient

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

We will use this $\vec{\nabla}$ in gradient, divergence & Curl

Basics of Gradient

- Gradient is vector quantity.
- Gradient is applied on scaler quantity.
- Gradient of function F can be calculated by,

$$Grad\,(F) = \vec{\nabla} F = \frac{\partial F}{\partial x}\hat{\imath} + \frac{\partial F}{\partial y}\hat{\jmath} + \frac{\partial F}{\partial z}\hat{k}$$

- It explains variation of function in x, y and z direction.

Basics of Divergence

- Divergence is scaler quantity.
- Divergence is applied on vector quantity.
- Divergence of function $\vec{F}$ can be calculated by,

$$Div\,(\vec{F}) = \vec{\nabla}.\vec{F} = \frac{\partial \vec{F}}{\partial x}.\hat{\imath} + \frac{\partial \vec{F}}{\partial y}.\hat{\jmath} + \frac{\partial \vec{F}}{\partial z}.\hat{k}$$

If $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

$$Div\,(\vec{F}) = \vec{\nabla}.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Basics of Gauss Divergence Theorem

- ❖ Divergence theorem explains relationship in between volume integration and surface integration.
- ❖ Divergence theorem also use to see the location of source and sink.
- ❖ Divergence theorem explains rate of change of function with respect to position.
- ❖ Divergence is flux density, it explains how much flux is entering or leaving the source or sink.

Proof of Stokes Theorem

- Curl of any function $\vec{P}$ can be calculated by

$$Curl\,(\vec{P}) = \vec{\nabla} \times \vec{P} = \lim_{\Delta S \to 0} \frac{\oint \vec{P}\,dl}{\Delta S}$$

- So further calculation can be done by

$$\Rightarrow \lim_{\Delta S \to 0} \vec{\nabla} \times \vec{P}\,\Delta S = \lim_{\Delta S \to 0} \oint \vec{P}\,dl$$

- So for total length

$$\oint \vec{P}.dl = \int \vec{\nabla} \times \vec{P}\,dS$$

Use of Stokes Theorem

- ❖ **It is used in applications of fluid mechanics**
- ❖ **It is used to understand electromagnetics**
- ❖ **It is used to understand flaw of fields (e.g. Gravitational fields, Electric field, Magnetic field etc)**
- ❖ **It is used in aerodynamics.**

Basics of Stokes Theorem

- Stokes theorem explains relationship in between line integration and surface integration.
- Stokes theorem is based on curl of function.
- Curl of function explains rotation of body at different position, means torque at the position.

→ If curl of the function is less then zero, then body will rotate in clockwise direction

The major goal is to solve Maxwell's equations and describe EM wave motion in the following media:

(1) Free space ($\sigma = 0, \varepsilon = \varepsilon_0, \mu = \mu_0$)

(2) Lossless dielectrics ($\sigma = 0, \varepsilon = \varepsilon_r \varepsilon_0, \mu = \mu_r \mu_0$ (or) $\sigma \ll \omega\varepsilon$)

(3) Lossy dielectrics ($\sigma \neq 0, \varepsilon = \varepsilon_r \varepsilon_0, \mu = \mu_r \mu_0$)

(4) Good conductor ($\sigma \simeq \infty, \varepsilon = \varepsilon_0, \mu = \mu_r \mu_0$ (or) $\sigma \gg \omega\varepsilon$)

- σ is the conductivity of the medium,
- ε is the permittivity of the medium
- μ is the permeability of the medium
- *ω* is the angular frequency of the wave.

A **homogenous medium** is one for which the quantities σ, ε and μ are constants throughout the medium.

The medium is **isotropic** if ε is constant, so that D and E have same direction everywhere.

Wave Propagation in free space : (source free wave equations)

Consider an electomagnetic wave propagating through free space.
The medium (free space) is sourceless ie., $\rho_V = 0$.

Free space contains no charges and hence no conduction current.

ie $\sigma = 0$, $J = \sigma E = 0$.

For free space the Maxwell's equations are as follows:

For free space the Maxwell's equations are as follows:

(i) $\nabla \times H = \frac{\partial D}{\partial t} = \varepsilon_0 \frac{\partial E}{\partial t}$ (J = 0 for free space)

(ii) $\nabla \times E = -\frac{\partial B}{\partial t} = -\mu_0 \frac{\partial H}{\partial t}$

(iii) $\nabla \cdot D = 0$ as $\rho_V = 0$ for free space

(iv) $\nabla \cdot B = 0$

Wave equation for Electric field:

$$\nabla \times H = \frac{\partial D}{\partial t} = \varepsilon_0 \frac{\partial E}{\partial t} \qquad (1)$$

Differentiating the above equation with respect to time:

$$\frac{\partial}{\partial t}(\nabla \times H) = \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \qquad (2)$$

∇ operator indicates differentiation with respect to space while $\frac{\partial}{\partial t}$ indicates differentiation with respect to time. Both are independent of each other and hence the operators can be interchanged.

Equation (2) is rewritten as

$$\nabla \times \frac{\partial H}{\partial t} = \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \qquad (3)$$

Taking the curl of the equation $\nabla \times E = -\frac{\partial B}{\partial t} = -\mu_0 \frac{\partial H}{\partial t}$

$$\nabla \times \nabla \times E = \nabla \times -\mu_0 \frac{\partial H}{\partial t} \qquad (4)$$

From vector identity

$$\nabla \times \nabla \times E = \nabla(\nabla . E) - \nabla^2 E \qquad (5)$$

Comparing equations (4) and (5) gives

$$\nabla(\nabla . E) - \nabla^2 E = -\nabla \times \mu_0 \frac{\partial H}{\partial t} = -\mu_0 \left(\nabla \times \frac{\partial H}{\partial t} \right) \qquad (6)$$

Substituting (3) in (6)

$$\nabla(\nabla . E) - \nabla^2 E = -\mu_0 \left(\varepsilon_0 \frac{\partial^2 E}{\partial t^2} \right) = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \qquad (7)$$

From equation $\nabla \cdot D = 0$

$$\nabla \cdot \varepsilon_0 E = 0$$

$$\nabla \cdot E = 0 \text{ since } \varepsilon_0 \neq 0 \qquad (8)$$

Substituting (8) in (7)

$$-\nabla^2 E = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \qquad (9)$$

This is the wave equation for a time varying electric field in free space conditions.

Wave equation for Magnetic field:

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu_0 \frac{\partial H}{\partial t} \qquad (1)$$

Differentiating the above equation with respect to time:

$$\frac{\partial}{\partial t}(\nabla \times E) = -\mu_0 \frac{\partial^2 H}{\partial t^2} \qquad (2)$$

∇ operator indicates differentiation with respect to space while $\frac{\partial}{\partial t}$ indicates differentiation with respect to time. Both are independent of each other and hence the operators can be interchanged.

Equation (2) is rewritten as

$$\nabla \times \frac{\partial E}{\partial t} = -\mu_0 \frac{\partial^2 H}{\partial t^2} \qquad (3)$$

Taking the curl of the equation $\nabla \times H = \frac{\partial D}{\partial t} = \varepsilon_0 \frac{\partial E}{\partial t}$

$$\nabla \times \nabla \times H = \nabla \times \varepsilon_0 \frac{\partial E}{\partial t} \qquad (4)$$

From vector identity

$$\nabla \times \nabla \times H = \nabla(\nabla . H) - \nabla^2 H \qquad (5)$$

Comparing equations (4) and (5) gives

$$\nabla(\nabla . H) - \nabla^2 H = \nabla \times \varepsilon_0 \frac{\partial E}{\partial t}$$

$$\nabla(\nabla . H) - \nabla^2 H = \varepsilon_0 \left(\nabla \times \frac{\partial E}{\partial t} \right) \qquad (6)$$

Substituting (3) in (6)

$$\nabla(\nabla . H) - \nabla^2 H = \varepsilon_0 \left(-\mu_0 \frac{\partial^2 H}{\partial t^2} \right) \qquad (7)$$

From equation $\nabla . B = 0$

$$\nabla . \mu_0 H = 0$$

$$\nabla . H = 0 \text{ since } \mu \neq 0 \qquad (8)$$

Substituting (8) in (7)

$$-\nabla^2 H = -\mu_0 \varepsilon_0 \frac{\partial^2 H}{\partial t^2}$$

$$\nabla^2 H = \mu_0 \varepsilon_0 \frac{\partial^2 H}{\partial t^2} \qquad (9)$$

This is the wave equation for a time varying magnetic field in free space conditions.

The wave equations for Electric field and magnetic field can be rewritten as:

$$\nabla^2 E - \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} = 0 \quad \text{(or)} \quad \nabla^2 E - \frac{1}{u^2}\frac{\partial^2 E}{\partial t^2} = 0$$

$$\nabla^2 H - \mu_0 \varepsilon_0 \frac{\partial^2 H}{\partial t^2} = 0 \quad \text{(or)} \quad \nabla^2 H - \frac{1}{u^2}\frac{\partial^2 H}{\partial t^2} = 0$$

Where $u = 1/\sqrt{\mu_0 \varepsilon_0} = 3 \times 10^8 \, m/s$ = Velocity of Propagation

The above equations are called homogenous vector wave equations

Using time harmonic Maxwells equations,

$$\nabla \times E = -j\omega\mu H,\ \nabla \times H = j\omega\varepsilon E,\ \nabla \cdot E = 0,\ \nabla \cdot H = 0 \text{ and}$$

$$\frac{\partial E}{\partial t} = j\omega E, \frac{\partial H}{\partial t} = j\omega H, \frac{\partial^2 E}{\partial t^2} = j^2\omega^2 E, \frac{\partial^2 H}{\partial t^2} = j^2\omega^2 H$$

The wave equations for Electric field and magnetic field can be rewritten as:

$$\nabla^2 E + \frac{\omega^2}{u^2} E = 0 \Rightarrow \nabla^2 E + K^2 E = 0$$

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Where, $K = \frac{\omega}{u}$ is the wave number.

The above equations are called homogenous vector Helmholtzs equations.