Module 2 PART-A

Electric Flux density, Gauss law & Divergence.

Electric Flux density

Michael Faraday's experimented, by taking a pair of Concentric metallic spheres, the outer one consisting of two hemispheres that could be firmly clamped together a dielectric material b/w the entire volume of the concentric spheres.

1. The inner sphere was given a +ve charge
2. Then hemisphere clamped together.
3. The outer sphere was discharged.

By separating the outer space, Faraday found that the total charge on outer sphere was equal in magnitude to the original charge placed on the inner sphere. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium as flux as displacement, displacement flux or Electric flux.

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{14}$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) = \underline{36.7^\circ}$$

$$\phi = \tan^{-1}(y/x) = \underline{63.43^\circ}$$

$$E_r = \vec{E}\cdot\hat{a}_r$$

$$= E_x \sin\theta\cos\phi + E_y \sin\theta\sin\phi + E_z\cos\theta$$

$$= \underline{554.7\ V/m}$$

$$E_\theta = E_x\cos\theta\cos\phi + E_y\cos\theta\sin\phi - E_z\sin\theta$$

$$= \underline{255}\ V/m$$

$$E_\phi = -E_x\sin\phi + E_y\sin\phi$$

$$= \underline{0.05} \simeq \underline{0 V/m}$$

$$E_{r\theta\phi} = \underline{554.7\,\hat{a}_r + 255\,\hat{a}_\theta}\ V/m$$

Faraday's experiment also showed that, a larger +ve charge on the inner sphere induce correspondingly a larger -ve charge on the outer sphere.

If electric flux is denoted by ψ & total charge on inner sphere by Q then,

$$\boxed{\psi = Q}$$

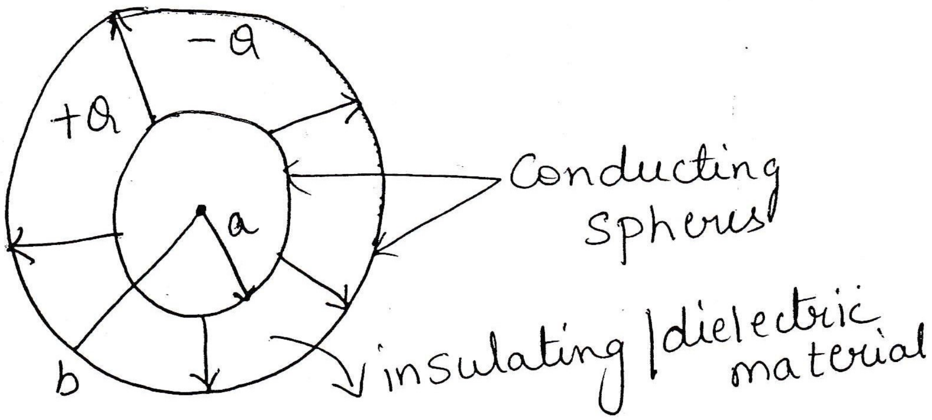

fig: electric flux in the region b/w a pair of charged concentric spheres.

at the surface of inner sphere, ψ Coulombs of electric flux are produced by charge Q $(=\psi)$ distributed uniformly over surface, are $4\pi a^2$.

∴ Electric Flux Density:

It is electric flux lines per unit area. (each flux line is due to one Coulomb). denoted by $\vec{D}$ & unit is C/m^2.

$$\vec{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\hat{a}_r$$

$$\vec{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\hat{a}_r$$

$$a \leq r \leq b$$

$$\boxed{\vec{D} = \frac{Q}{4\pi r^2}\hat{a}_r}$$

W.K.T the radial electric field intensity

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r$$

In free space,

$$\boxed{\vec{D} = \varepsilon_0\vec{E}}$$

1) Find $\vec{D}$ in the region about a uniform line charge of 8 nc/m lying along z-axis in free space, at $\rho = 3$m.

Soln:

$$\vec{E} = \frac{\rho_l \, \hat{a}_\rho}{2\pi\varepsilon_0 \rho} = \frac{8\times10^{-9}}{2\pi\varepsilon_0 \times 3}\hat{a}_\rho = 47.9\,\hat{a}_\rho \text{ V/m}$$

$$\vec{D} = \frac{\rho_l}{2\pi\rho}\hat{a}_\rho = \varepsilon_0 \vec{E} = 0.424\,\hat{a}_\rho \text{ nc/m}^2$$

2) given a 60 μc point charge located at the origin, find the total electric flux passing through (a) that portion of the sphere $r = 26$ cm bounded by $0 < \theta < \pi/2$ & $0 < \phi < \pi/2$

$$\psi = \oint_S \vec{D}\cdot\vec{ds}$$

$$= \oint_S \frac{Q}{4\pi r^2}\hat{a}_r \cdot r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$$

$$= \frac{Q}{4\pi r^2}\cdot r^2 \left[-\cos\theta\right]_0^{\pi/2}\left[\phi\right]_0^{\pi/2}$$

$$= \frac{Q}{4\pi}\times\frac{\pi}{2}\times 1 = \frac{Q}{8}$$

$$\psi = \frac{Q}{8} = 7.5\,\mu c$$

(b) the closed surface defined by $\rho = 26cm$ & $z = \pm 26cm$

$\Rightarrow$ It covers full sphere.

$\therefore \psi = Q = 60 \mu C$

(c) the plane $z = 26cm$

It covers half of sphere.

$\psi = \frac{Q}{2} = 30 \mu C$

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3) Calculate $\vec{D}$ in rectangular Co-ordinates at point $P(2, -3, 6)$ produced by

(a) A point charge $Q_A = 55 mC$ at $Q(-2, 3, -6)$

(b) a uniform line charge $\rho_{LB} = 20 mC/m$ on x-axis.

Soln $\vec{D} = \vec{D}_1$

$\vec{D}_1 \rightarrow$ Electric flux density due to point charge.

$\vec{D}_2 \rightarrow$ " " " " line charge

$$\vec{D}_1 = \frac{Q_A}{4\pi\epsilon_0 |\vec{R}|^2} \hat{a}_R = \frac{55 \times 10^{-3}}{4\pi |\vec{R}|^2} \hat{a}_R$$

$$\vec{R} = 4\hat{a}_x - 6\hat{a}_y + 12\hat{a}_z$$

$$|\vec{R}| = \underline{14}$$

$$\vec{D}_1 = 1.59 \times 10^{-6} (4\hat{a}_x - 6\hat{a}_y + 12\hat{a}_z)$$

$$= \underline{6.38\hat{a}_x - 9.57\hat{a}_y + 19.14\hat{a}_z} \ \mu C/m^2$$

(b) $$\vec{D}_2 = \frac{\rho_l \hat{a}_\rho}{2\pi\rho}$$

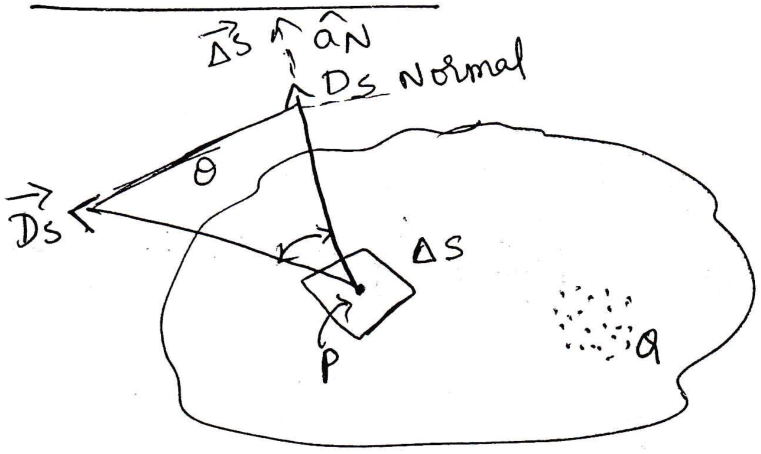
$$\vec{R} = -3\hat{a}_y + 6\hat{a}_z$$

$$\rho = |\vec{R}| = \underline{6.71}$$

$$\vec{D}_2 = \frac{20 \times 10^{-3}}{2\pi \times 6.71} \frac{(-3\hat{a}_y + 6\hat{a}_z)}{6.71} = 7.069 \times 10^{-5} (-3\hat{a}_y + 6\hat{a}_z)$$

$$\vec{D}_2 = \underline{-212\hat{a}_y + 424\hat{a}_z} \ \mu C/m^2$$

Gauss's Law



Cloud of Point charges as in above figure, Kept in closed surface. If the total Charge is Q. Then Q Coulombs of electric flux will pass through the enclosing surface.

At any point P Consider an incremental element of surface ΔS. The flux Crossing ΔS is product of normal Component of $\vec{D_s}$ & $\vec{\Delta S}$.

$$\Delta \psi = \vec{D_s} \cdot \vec{\Delta S}$$

the total flux passing through the closed surface.

$$\psi = \int d\psi = \oint_{\substack{closed \\ surface}} \vec{D_s} \cdot \vec{ds}$$

$$\therefore \boxed{\psi = \oint_S \vec{D} \cdot \vec{ds} = Q}$$ charge enclosed

Gauss's law

line charge,

$$Q = \int \rho_l \, dl$$

Surface charge

$$Q = \int_S \rho_s \, ds$$

Volume distribution

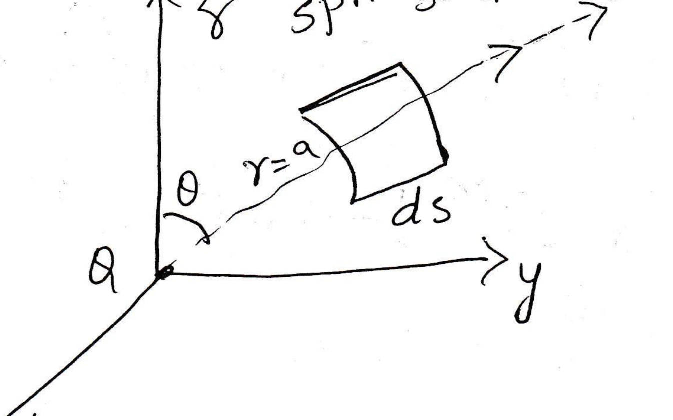
$$Q = \int_{vol} \rho_v \, dv$$

$$\boxed{\oint_S \vec{D_s} \cdot \vec{ds} = \int_{vol} \rho_v \, dv}$$

Gauss law states that

" The total electric flux passing through any closed surface is equal to total charge enclosed".

To prove,



$$\vec{D} = \frac{Q}{4\pi r^2} \hat{a}_r$$

$$d\psi = \vec{D} \cdot \vec{ds}$$

$$= \frac{Q}{4\pi r^2} \hat{a}_r \cdot \vec{ds}$$

if $\vec{ds_r} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{a_r}$

$$d\psi = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$\psi = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$\boxed{\psi = Q}$$

1) given electric flux density $\vec{D} = 0.3r^2 \hat{a_r}$ nc/m^2 in free space.

(a) Find $\vec{E}$ at $P(r=2, \theta=25°, \phi=90°)$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{0.3 \times 2^2 \times 10^{-9} \hat{a_r}}{8.854 \times 10^{-12}} = \underline{135.5 \hat{a_r} \, V/m}$$

(b) Find the total charge within the sphere $r=3$

$$Q = \oint_s \vec{D} \cdot \vec{ds}$$

$$= \int_\theta \int_\phi 0.3r^2 \hat{a_r} \cdot r^2 \sin\theta \, d\theta \, d\phi \, \hat{a_r} \times 10^{-9}$$

$$= 0.3r^4 \left. \phi \right]_0^{2\pi} \left[-\cos\theta \right]_0^{\pi} \times 10^{-9}$$

$$= 0.3r^4 (2\pi)(2) \times 10^{-9}$$

$= 0.3(3)^4(4\pi) \times 10^{-9}$

$= 305.4 \times 10^{-9}$

$= \underline{\underline{305.4 nC}}$

(c) Find the total electric flux leaving the sphere $r=4$.

$Q = 0.3r^4 4\pi \times 10^{-9}$

$= 0.3 \times 4^4 \times 4\pi \times 10^{-9}$

$= \underline{\underline{965.1 nC}}$

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2) A line charge density of 24nC/m is located in free space on the line $y=1, z=2$

(a) Find $\vec{E}$ at $P(6,-1,3)$

(b) What point charge Q_A should be located at $Q(-3,4,1)$ to cause E_y to be equal to zero at P?.

Soln:

(a) $\vec{R} = -2\hat{a}_y + \hat{a}_z$

$|\vec{R}| = \sqrt{5}$

$$E_1 = \frac{24 \times 10^{-9}}{2\pi \varepsilon_0 \times \sqrt{5}} \frac{(-2\hat{a}_y + \hat{a}_z)}{\sqrt{2}} = -172.76\hat{a}_y + 86.379\hat{a}_z \text{ V/m}$$

(b) $\vec{r}_{AP} = 9\hat{a}_x - 5\hat{a}_y + 2\hat{a}_z$

$|\vec{r}_{AP}| = \sqrt{110}$

$$\vec{E}_Q = \frac{Q_A}{4\pi\varepsilon_0} \frac{(9\hat{a}_x - 5\hat{a}_y + 2\hat{a}_z)}{(110)^{1.5}}$$

$$\frac{Q_A}{4\pi\varepsilon_0}\left(\frac{-5\hat{a}_y}{(110)^{1.5}}\right) = +\underline{172.76}\hat{a}_y$$

$$Q_A = \underline{\underline{-4.43\,\mu C}}$$

Divergence:

The divergence of any vector $\vec{A}$,

$$\text{Div of } \vec{A} = \nabla \cdot \vec{A} = \underset{\Delta V \to 0}{Lt} \frac{\oint_S \vec{A} \cdot \vec{ds}}{\Delta V}$$

lll^{ly}

$$\nabla \cdot \vec{D} = \underset{\Delta V \to 0}{Lt} \frac{\oint_S \vec{D} \cdot \vec{ds}}{\Delta V}$$

$$= \underset{\Delta V \to 0}{Lt} \frac{Q}{\Delta V}$$

$$\Rightarrow \boxed{\nabla \cdot \vec{D} = \underset{\Delta V \to 0}{Lt} \frac{\oint \vec{D} \cdot \vec{ds}}{\Delta V} = \rho_V}$$

$$\text{div } \vec{D} = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z} \quad (\text{in Cartesian})$$

The divergence of vector flux density $\vec{A}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

A +ve divergence for any vector quantity indicates a source of that vector quantity at that point.

Ily a -ve divergence indicates a sink.

$\boxed{\nabla \cdot \vec{D} = 0}$ called Solenoidal.

$$\nabla \cdot \vec{D} = \text{div } \vec{D} = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z} \text{ [Rectangular]}$$

$$= \frac{1}{\rho} \frac{\delta}{\delta \rho}(\rho D_\rho) + \frac{1}{\rho} \frac{\delta D_\phi}{\delta \phi} + \frac{\delta D_z}{\delta z} \text{ [cylindrical]}$$

$$= \frac{1}{r^2} \frac{\delta}{\delta r}(r^2 D_r) + \frac{1}{r\sin\theta} \frac{\delta}{\delta \theta}(\sin\theta D_\theta)$$

$$+ \frac{1}{r\sin\theta} \frac{\delta D_\phi}{\delta \phi} \text{ [Spherical]}$$

(1) Find $\nabla \cdot \vec{D}$ at origin if $\vec{D} = e^{-x}\sin y\,\hat{a}_x - e^{-x}\cos y\,\hat{a}_y + 2z\hat{a}_z$

$$\nabla \cdot \vec{D} = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$$

$$= -e^{-x}\sin y + e^{-x}\sin y + 2 = 2 \text{ C/m}^3$$

(2) Find $\nabla \cdot \vec{D}$

(a) $\vec{D} = (2xyz - y^2)\hat{a}_x + (x^2 z - 2x)\hat{a}_y + (x^2 y)\hat{a}_z \; c/m^2$

at $P_A\,(2, 3, -1)$.

$$\nabla \cdot \vec{D} = (2yz - 0) + 0 + 0$$

$$\nabla \cdot \vec{D} \big|_{P_A} = 2 \times 3(-1) - 0 = \underline{\underline{-6 \; c/m^3}}$$

(b) $\vec{D} = 2\rho z^2 \sin^2\phi\, \hat{a}_\rho + \rho z^2 \sin 2\phi\, \hat{a}_\phi + 2\rho^2 z \sin^2\phi\, \hat{a}_z \; c/m^2$

at $P_B(\rho = 2,\ \phi = 110^\circ,\ z = -1)$

soln

$$\nabla \cdot \vec{D} = \frac{1}{\rho}\frac{\delta}{\delta \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\delta D_\phi}{\delta \phi} + \frac{\delta D_z}{\delta z}$$

$$= \frac{1}{\rho}\frac{\delta}{\delta\rho}(\rho \cdot 2\rho z^2 \sin^2\phi) + \frac{1}{\rho}\frac{\delta}{\delta\phi}(\rho z^2 \sin 2\phi) + \frac{\delta}{\delta z}(2\rho^2 z \sin^2\phi)$$

$$= \frac{1}{\rho} \times 4\rho z^2 \sin^2\phi + \frac{1}{\rho} \times \rho z^2 \times 2\cos 2\phi + 2\rho^2 \sin^2\phi$$

$$= 4z^2 \sin^2\phi + 2z^2 \cos 2\phi + 2\rho^2 \sin^2\phi$$

$$= \underline{\underline{9.06 \; c/m^3}}$$

(c) $\vec{D} = 2r\sin\theta\cos\phi\,\hat{a}_r + r\cos\theta\cos\phi\,\hat{a}_\theta - r\sin\phi\,\hat{a}_\phi$ C/m^2

at P_c $(r = 1.5,\ \theta = 30^\circ,\ \phi = 50^\circ)$

Soln

$$\nabla\cdot\vec{D} = \frac{1}{r^2}\frac{\delta}{\delta r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\delta}{\delta\theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\cdot\frac{\delta D_\phi}{\delta\phi}$$

$$= \frac{1}{r^2}\frac{\delta}{\delta r}(r^2\, 2r\sin\theta\cos\phi) + \frac{1}{r\sin\theta}\cdot\frac{\delta}{\delta\theta}(\sin\theta\, r\cos\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\delta}{\delta\phi}(-r\sin\phi)$$

$$= \frac{2}{r^2}\cdot 3r^2\sin\theta\cos\phi + \frac{1}{r\sin\theta}\,2\cos 2\theta\cos\phi - \frac{\cos\phi}{\sin\theta}$$

$$\nabla\cdot\vec{D}\Big|_{P_c} = \underline{\underline{1.29\,C/m^3}}$$

Maxwell's First Equation (Electrostatics)

From Gauss's law

$$\oint_S \vec{D} \cdot \vec{ds} = Q$$

W.K.T

$$\text{div } \vec{D} = \underset{\Delta V \to 0}{Lt} \frac{\oint_S \vec{D} \cdot \vec{ds}}{\Delta V} = \underset{\Delta V \to 0}{Lt} \frac{Q}{\Delta V}$$

As the Volume shrinks to zero,

$$\underset{\Delta V \to 0}{Lt} \frac{\oint_S \vec{D} \cdot \vec{ds}}{\Delta V} = \underset{\Delta V \to 0}{Lt} \frac{Q}{\Delta V}$$

Hence $\boxed{\nabla \cdot \vec{D} = \rho_v}$ called as Point form of Gauss's law

the above equation is the first Maxwell's eqn for Electrostatic & Steady magnetic field.

It States that electric flux per unit volume leaving a Vanishingly small unit volume is exactly equal to the volume charge density there.

1) Determine an expression for the volume charge density associated with each $\vec{D}$ field.

(a) $\vec{D} = \frac{4xy}{z}\hat{a}_x + \frac{2x^2}{z}\hat{a}_y - \frac{2x^2y}{z^2}$

$$\rho_v = \nabla \cdot \vec{D} = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$$

$$= \frac{4y}{z} + 0 - 2x^2y\,\frac{(-2)}{z^3} = \frac{4y}{z} + \frac{4x^2y}{z^3}$$

$$= \frac{4y}{z^3}(x^2+z^2)$$

(b) $\vec{D} = z\sin\phi\,\hat{a}_\rho + z\cos\phi\,\hat{a}_\phi + \rho\sin\phi\,\hat{a}_z$

$$\rho_v = \nabla \cdot \vec{D} = \frac{1}{\rho}\frac{\delta D_\rho}{\delta \rho} + \frac{1}{\rho}\frac{\delta D_\phi}{\delta \phi} + \frac{\delta D_z}{\delta z}$$

$$= \frac{1}{\rho} z \cancel{\sin\phi} + \frac{1}{\rho} z(\cancel{-\sin\phi}) + 0$$

$$= \underline{\underline{0}}$$

(c) $\vec{D} = \sin\theta \sin\phi \, \hat{a}_r + \cos\theta \sin\phi \, \hat{a}_\phi + \cos\phi \, \hat{a}_\phi$

$$\rho_v = \nabla \cdot \vec{D} = \frac{1}{r^2} \frac{\delta}{\delta r}(r^2 D_r) + \frac{1}{r\sin\theta} \frac{\delta}{\delta\theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta} \frac{\delta D_\phi}{\delta\phi}$$

$$= \frac{1}{r^2} 2r \sin\theta \sin\phi + \frac{1}{r\sin\theta} \frac{2\cos 2\theta \sin\phi}{2} + \frac{1}{r\sin\theta}(-\sin\phi)$$

$$= \frac{2}{r} \sin\theta \sin\phi + \frac{1}{r} \frac{\cos 2\theta \sin\phi}{\sin\theta} + \frac{-\sin\phi}{r\sin\theta}$$

$$= \frac{\sin\phi}{r} \left[2\sin\theta + \frac{\cos 2\theta}{\sin\theta} - \frac{1}{\sin\theta} \right]$$

$$= \frac{\sin\phi}{r} \left[\frac{2\sin^2\theta - 1 + \cos 2\theta}{\sin\theta} \right]$$

$$= \frac{\sin\phi}{r} \left[\frac{2\sin^2\theta - 1 + 2\cos^2\theta - 1}{\sin\theta} \right]$$

$$= \frac{\sin\phi}{r} \left[\frac{2-2}{\sin\theta} \right] = \underline{\underline{0}}$$

Gauss Divergence theorem

From Gauss law we can write

$$\oint \vec{D} \cdot \vec{ds} = Q$$

& $Q = \int_{vol} \rho_v \, dv$

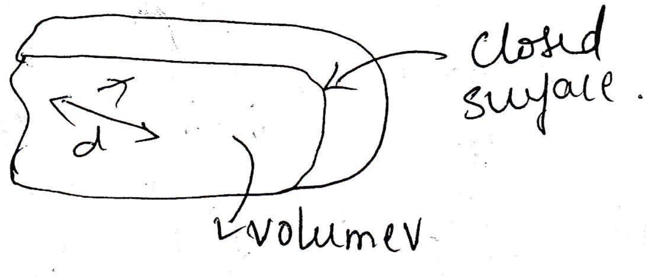
from Point form of Gauss law

$$\nabla \cdot \vec{D} = \rho_v$$

$$\oint \vec{D} \cdot \vec{ds} = Q = \int_{vol} \rho_v \, dv$$

$$\boxed{\oint \vec{D} \cdot \vec{ds} = \int_{vol} \nabla \cdot \vec{D} \, dv}$$

It states that the total flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume.



i) Evaluate both side of Gauss Divergence theorem for $\vec{D} = 2xy\hat{a}_x + x^2\hat{a}_y$ C/m^2. & the rectangular parallelpiped formed by Planes $x = 0$ & 1, $y = 0$ & 2 & $z = 0$ & 3.

soln

D is $\parallel^{el}$ to surfaces at $z = 0$ & $z = 3$

$$\vec{D} \cdot \vec{ds} = 0$$

for remaining four surfaces.

$$\oint_s \vec{D} \cdot \vec{ds} = \int_0^3 \int_0^2 (\vec{D})_{x=0} (-dy\,dz\,\hat{a}_x) + \int_0^3 \int_0^2 (D)_{x=1}\, dy$$

$$+ \int_0^3 \int_0^1 (\vec{D})_{y=0} (-dx\,dz)\hat{a}_y + \int_0^3 \int_0^1 (\vec{D})_{y=2}\, dx\,dy\,\hat{a}_y$$

$$= -\int_0^3 \int_0^2 2xy\,dy\,dz \Big|_{x=0} + \int_0^3 \int_0^2 2xy\,dy\,dz \Big|_{x=1}$$

$$- \int_0^3 \int_0^1 x^2\,dx\,dz \Big|_{y=0} + \int_0^3 \int_0^1 x^2\,dx\,dz \Big|_{y=2}$$

$$= -\left[2x\frac{y^2}{2}\right]_0^2 . \, z\Big]_0^3 + \cancel{2}x\frac{y^2}{\cancel{2}}\Big]_0^2 \, 3\Big]_0^3$$

$$- \frac{x^3}{3}\Big]_0^1 \, z\Big]_0^3 + \frac{x^3}{3}\Big]_0^1 \, z\Big]_0^3$$

$= \underline{\underline{12}}$

$$\nabla \cdot \vec{D} = \frac{\delta}{\delta x}(2xy) + \frac{\delta}{\delta y}(x^2) = 2y.$$

$$\int_{vol} \nabla \cdot \vec{D}\, dv = \int_0^3 \int_0^2 \int_0^1 2y \, dx\,dy\,dz$$

$$= \left.\frac{2y^2}{2}\right]_0^2 \left.x\right]_0^1 \left.z\right]_0^3$$

$= \underline{\underline{12}}$ Hence proved.

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2) Find the total Charge in a volume defined by 6 planes for which $1 \leq x \leq 2$, $2 \leq y \leq 3$, $3 \leq z \leq 4$, if $\vec{D} = 4x\hat{a_x} + 3y^2\hat{a_y} + 2z^2\hat{a_z}\, C/m^2$

Soln

$$\rho_v = \nabla \cdot \vec{D}$$

$$Q = \int_{vol} \rho_v \, dv$$

$$\rho_v = \nabla \cdot \vec{D}$$

$$= \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z} = (4 + 6y + 4z)\, C/m^3$$

$$Q = \int_{x=1}^{2} \int_{y=2}^{3} \int_{z=3}^{4} (4 + 6y + 4z)\, dx\,dy\,dz$$

$$= 4x\Big]_1^2 \, y\Big]_2^3 \, z\Big]_3^4 + 6x\Big]_1^2 \, \frac{y^2}{2}\Big]_2^3 \, z\Big]_3^4$$

$$+ 4x\Big]_1^2 \, y\Big]_2^3 \, \frac{z^2}{2}\Big]_3^4$$

$$= 4(1)(1)(1) + 6(1)\frac{1}{2}(9-4)\times 1 + 4\times 1\times 1\times \frac{1}{2}(4^2-3^2)$$

$$= 4+15+14 = \underline{\underline{33C}}$$

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3) If $\vec{G} = 5r\sin^2\theta\cos^2\phi\,\hat{a}_r$. Evaluate both side of divergence theorem for the region $r \leq 2$.

$$Q = \oint_S \vec{D}\cdot d\vec{s}$$

$\sin 3\theta = 3\sin\theta - 4\sin^3$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r r^2 \sin\theta \, d\theta \, d\phi$$

$$= 5\int_\phi\int_\theta r^3 \sin^3\theta \cos^2\phi \, d\theta \, d\phi$$

$$= 5\times 2^3 \times \frac{1}{4}\left[\frac{\cos 3\theta}{3} - 3\cos\theta\right]_0^\pi \frac{1}{2}\left(\phi + \frac{\sin 2\phi}{2}\right)_0^{2\pi}$$

$$= \underline{\underline{167.6C}}$$

$$\rho_v = \nabla \cdot \vec{D}$$

$$= \frac{1}{r^2}\frac{\delta}{\delta v}(r^2 D_r)$$

$$= \frac{1}{r^2}\frac{\delta}{\delta r}(5r^3 \sin^2\theta \cos^2\phi)$$

$$= \frac{1}{r^2} \times 15 r^2 \sin^2\theta \cos^2\phi$$

$$= 15 \sin^2\theta \cos^2\phi$$

$$Q = \int_{vol} \nabla \cdot \vec{D}\, dv$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{2} 15 \sin^2\theta \cos^2\phi\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 15\left[\frac{r^3}{3}\right]_0^2 \times \frac{1}{4}\left[\frac{\cos 3\theta}{3} - 3\cos\theta\right]_0^{\pi} \times \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\phi\right]_0^{2\pi}$$

$$= \underline{\underline{167.47C}}$$

Module-2

Part B: Energy And Potential

Books Referred.

1) "Engineering Electromagnetics," William H Hayt Jr and John A Buck, Tata Mc-Graw Hill, 7th edition 2006.
2) "Electromagnetics with Applications", John Krauss and Daniel A Fleisch, Mc Graw-Hill, 5th edition 1999.
3) "Field and wave Electromagnetics," David K Cheng, Pearson Education Asia, 2nd edition - 1989.

Topics Covered

- Energy Expended in moving a point charge in an Electric field.
- The line integral
- Definition of potential difference & Potential
- The potential field of a point charge & system of charges.
- Potential gradient.
- Energy density in an electrostatic field.
- Current & current density.
- Continuity Equation.

Energy Expended in moving a point charge in an Electric field.

The electric field intensity is defined as the force on a unit test charge at that point, where the value of vector field is found.

If a test charge is moved against the electric field, there is expenditure of energy or do work.

Or if a charge is moved in the direction of field, energy expenditure turns out to be -ve, no work, the field does.

If a charge Q is moved a distance $\vec{dL}$ in an electric field $\vec{E}$. The force on Q arising from the electric field is

$$\boxed{\vec{F_E} = Q\vec{E}}$$

The component of this force in the direction $\vec{dL}$ which must overcome is

$$\vec{F_{EL}} = F \cdot \hat{a}_L = Q\vec{E} \cdot \hat{a}_L$$

where $\hat{a}_L$ = unit vector in the direction of $\vec{dL}$.

The force which we must apply is equal & opposite to the force associated with the field,

$$\boxed{\vec{F}_{appl} = -Q\vec{E} \cdot \hat{a}_L}$$

∴ the differential work done by external source moving $Q = -Q\vec{E} \cdot \hat{a}_L \, dL$

$$= -Q\vec{E} \cdot \vec{dL}$$

or $\boxed{dW = -Q\vec{E}\cdot\vec{dL}}$ we have replaced $\hat{a}_L dl$ by $\vec{dL}$

The work required to move the charge a finite distance must be determined by integrating " work done by electric field in moving a charge".

$$\boxed{W = -Q\int_{initial}^{final} \vec{E}\cdot\vec{dl}} \text{ Joules}$$

Problems:

1) A non uniform field, $\vec{E} = y\hat{a}_x + x\hat{a}_y + 2\hat{a}_z$ Determine the work expended in carrying 2C from B (1, 0, 1) to A (0.8, 0.6, 1) along the shorter (a) arc of a circle, $x^2+y^2=1$, $z=1$.

Soln

$$W = -Q\int_B^A \vec{E}\,\vec{dl}$$

$$= -2\int_B^A (y\hat{a}_x + x\hat{a}_y + 2\hat{a}_z)\cdot(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z)$$

$$= -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^1 dz$$

$$= -2\int_1^{0.8}\sqrt{1-x^2}\,dx - 2\int_0^{0.6}\sqrt{1-y^2}\,dy - 0$$

$$= -\left[x\sqrt{1-x^2} + \sin^{-1}x\right]_1^{0.8} - \left[y\sqrt{1-y^2} + \sin^{-1}y\right]_0^{0.6}$$

$$= -(0.48 + 0.927 - 0 - 1.571) - (0.48 + 0.644 - 0 - 0)$$

$$= \underline{-0.965}$$

(b) use a straight line path from B to A.

$$y - y_B = \frac{y_A - y_B}{x_A - x_B}(x - x_B)$$

$$\Rightarrow \quad y = -3(x-1)$$

$$z - z_B = \frac{z_A - z_B}{y_A - y_B}(y - y_B)$$

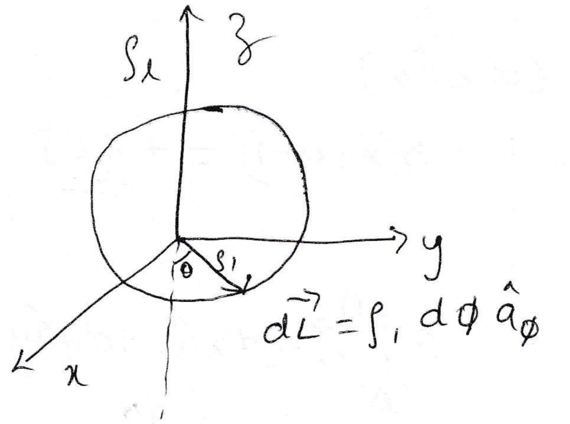
$$\Rightarrow \quad \underline{z = 1}$$

$$W = -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^1 dz$$

$$= 6\int_1^{0.8}(x-1)\,dx - 2\int_0^{0.6}\left(1 - \frac{y}{3}\right)dy$$

$$= \underline{-0.965\ J}$$

2) Find the work done in carrying the +ve charge Q about a circular path of radius ρ_b centred at the line charge.



the $\vec{dL}$ is cylindrical

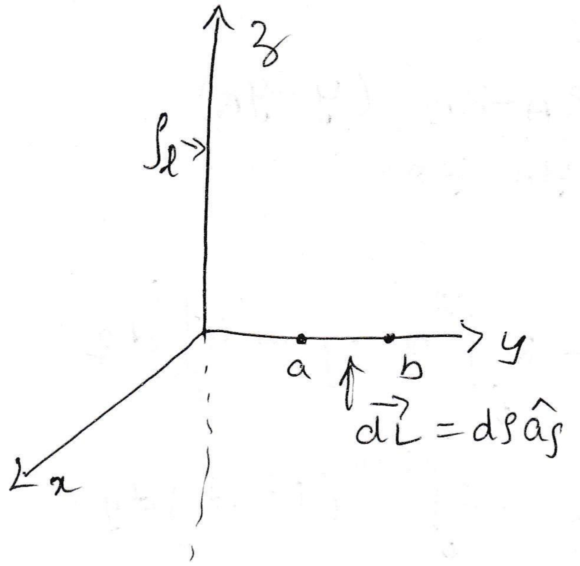
$$\vec{dL} = \rho_1\,d\phi\,\hat{a}_\phi$$

$$W = -Q\int_{ini}^{fin} \frac{\rho_l}{2\pi\varepsilon_0\rho_1}\hat{a}_\rho \cdot \rho_1 d\phi\hat{a}_\phi$$

$$W = -Q\int_0^{2\pi} \frac{\rho_l}{2\pi\varepsilon_0\rho}\,d\phi\,(\hat{a}_\rho - \hat{a}_\phi)$$

$$\underline{W = 0}$$

③ Find the work done in carrying a +ve charge Q, along the radial path $\rho = a$ to $\rho = b$ with line charge.



$$W = -Q \int_{ini}^{fin} \vec{E} \cdot \vec{dL}$$

$$W = -Q \int_a^b \frac{\rho_L}{2\pi\varepsilon_0 \rho}\, d\rho$$

$$\boxed{W = \frac{-Q\rho_l}{2\pi\varepsilon_0} \ln \frac{b}{a} \ J}$$

---

4) Calculate the work done in moving a 4C charge from B(1, 0, 0) to A(0, 2, 0) along the path $y = 2 - 2x$, $z = 0$ in the field $\vec{E}$.

(a) $5\hat{a}_x$ V/m

$$W = -Q \int_{ini}^{final} \vec{E} \cdot \vec{dL} = -Q \int_B^A \vec{E} \cdot \vec{dL}$$

$$= -4 \int_1^0 5\hat{a}_x \cdot (dx\, \hat{a}_x)$$

$$= -4 \times 5x \Big]_1^0 = -4 \times 5 \times (0-1) = \underline{\underline{+20 J}}$$

(b) $5x\hat{a}_x$ V/m

$$W = -Q \int_{ini}^{final} \vec{E} \cdot \vec{dL} = -Q \int_B^A \vec{E} \cdot (dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z)$$

$$= -4 \left[\int_1^0 5x\, dx \right] = -4 \times 5 \cdot \frac{x^2}{2} \Big]_1^0$$

$$= -10[0-1] = \underline{\underline{+10 J}}$$

c) $5x\hat{a}_x + 5y\hat{a}_y$ V/m

$$W = -Q\int_{ini}^{fin} \vec{E}\cdot\vec{dl} = -Q\int_B^A \vec{E}\cdot(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z)$$

$$= -4\left[\int_1^0 5x\,dx + \int_0^2 5y\,dy\right]$$

$$= -4\left[5\frac{x^2}{2}\Big]_1^0 + 5\frac{y^2}{2}\Big]_0^2\right]$$

$$= -4\left[\frac{5}{2}(0-1) + \frac{5}{2}(4-0)\right] = -4\times(10-5/2)$$

$$= -4\times\frac{15}{2} = -30\text{ J}$$

---

4) Let $\vec{E} = y\,\hat{a}_x$ V/m at a certain instant of time Calculate the work required to move a 3C charge from (1, 3, 5) to (2, 0, 3) along the straight line segments joining.

(1, 3, 5) to (2, 3, 5) to (2, 0, 5) to (2, 0, 3)

$$W_1 = -Q\int_{(1,3,5)}^{(2,3,5)} \vec{E}\cdot\vec{dl} = -3y\,x\Big]_1^2 = -3\times3 = -9\text{J}$$

$$W_2 = -Q\int_{(2,3,5)}^{(2,0,5)} \vec{E}\cdot\vec{dl} = -3y\,(x)_2^2 = -3y\,(2-2) = 0$$

$$W_3 = -Q\int_{(2,0,5)}^{(2,0,3)} \vec{E}\cdot\vec{dl} = -3y\,(x)_2^2 = 0$$

$\therefore$ total $W = W_1 + W_2 + W_3$

$$= -9\text{J}$$

5) Given the field $\vec{E} = \frac{1}{z^2}(8xyz\,\hat{a}_x + 4x^2z\,\hat{a}_y - 4x^2y\,\hat{a}_z)$ V/m find the differential amount of work done in moving 6nc charge a distance of 2 um starting at P(2, -2, 3) & in direction $\hat{a}_L$

(a) $-\frac{6}{7}\hat{a}_x + \frac{3}{7}\hat{a}_y + \frac{2}{7}\hat{a}_z$

$$dw = -q\,\vec{E}\cdot\vec{dL}$$

$$= -q\,\vec{E}\cdot dL\,\hat{a}_L$$

$$= -6\times10^{-9}\times2\times10^{-6}\left\{\left(\frac{1}{z^2}(8xyz)\left(\frac{-6}{7}\right) + \frac{4x^2z}{z^2}\left(\frac{3}{7}\right) - \frac{4x^2y}{z^2}\left(\frac{2}{7}\right)\right\}$$

$$= -12\times10^{-15}\{9.14 + 2.29 + 1.02\}$$

$$= \underline{-149.3\ fJ}$$

$$= \underline{-149.3\times10^{-15}\ J}$$

(b) $\hat{a}_L = \frac{3}{7}\hat{a}_x + \frac{6}{7}\hat{a}_y$

$$\underline{dw = 0}$$

The Line Integral

Consider the charged moved from initial position 'B' to the final position 'A' against the electric field $\vec{E}$, then the work done is given by

$$W = -Q\int_B^A \vec{E}\cdot\vec{dl}$$

this is called line integral where $\vec{E}\cdot\vec{dl}$ gives the component of $\vec{E}$ along the direction $\vec{dl}$.

line integral is basically a summation & accurate result obtained when the no of segments become infinite.

Consider an uniform electric field $\vec{E}$ the charge is moved from B to A along the path. The path is divided into six segments $\Delta L_1, \Delta L_2$, ---- ΔL_6 & the components of $\vec{E}$ along each segment are denoted by E_{L1}, E_{L2}, --- E_{L6}. The work involved in moving a charge Q from B to A is

$$W = -Q(\vec{E}_{L1}\,\Delta L_1 + \vec{E}_{L2}\,\Delta \vec{L}_2 + \cdots E_{L6}\,\Delta L_6)$$

using vector notation,

$$W = -Q(\vec{E}_1\cdot\Delta L_1 + \vec{E}_2\cdot\Delta L_2 + \cdots \vec{E}_6\cdot\Delta L_6)$$

assuming a uniform field.

$$W = -Q\,\vec{E}\cdot\vec{L}_{BA}$$

$$W = -Q\int_B^A \vec{E}\cdot\vec{dl} \quad \text{(uniform field)}$$

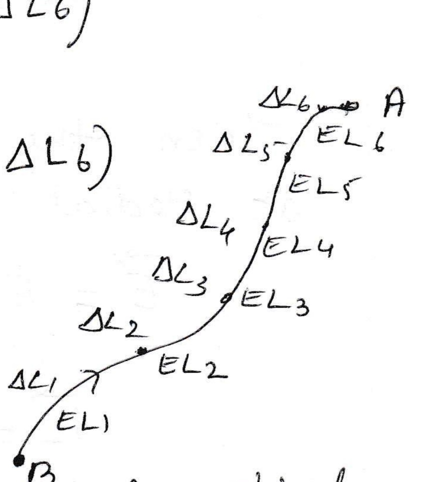


fig: A graphical representation of a line integral in a uniform field.

this is true for non-uniform field also.

Potential Difference and Potential

Definition: Potential difference V is the work done (by an external source) in moving a unit +ve charge from one point to another in an electric field.

$$\text{Potential difference} = V = -\int_{ini}^{fin} \vec{E}\cdot\vec{dl} \quad J/c \text{ or } V$$

the potential difference between points A & B

$$V_{AB} = -\int_{B}^{A} \vec{E}\cdot\vec{dl}$$

In the line charge, work done in carrying a +ve charge in radial direction from $\rho = b$ & $\rho = a$ is

$$W = \frac{Q\rho_l}{2\pi\varepsilon_0} \ln \frac{b}{a}$$

$$V_{ab} = \frac{W}{Q} = \frac{\rho_l}{2\pi\varepsilon_0} \ln \frac{b}{a}$$

Then the potential difference b/w points A & B at radial distance r_A & r_B from a point charge

$$\vec{E} = E_r \hat{a}_r = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{a}_r$$

$$\vec{dL} = dr\, \hat{a}_r$$

$$V_{AB} = -\int_{B}^{A} \vec{E}\cdot\vec{dl} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2} dr$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

If Potential at point A is V_A & at B is V_B

then $\boxed{V_{AB} = V_A - V_B}$

V_A & V_B have the Same zero reference Point.

Problems:

1) An electric field is expressed in rectangular Co-ordinates
$\vec{E} = 6x^2\hat{a}_x + 6y\hat{a}_y + 4\hat{a}_z$ V/m.

find (a) V_{MN} if Points M & N are specified by $M(2,6,-1)$ & $N(-3,-3,2)$.

$$V_{MN} = -\int_N^M \vec{E}\cdot\vec{dL}$$

$$= -\int_N^M (6x^2\hat{a}_x + 6y\hat{a}_y + 4\hat{a}_z)\cdot(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z)$$

$$= -\int_{-3}^{2} 6x^2\,dx - \int_{-3}^{6} 6y\,dy - \int_{2}^{-1} 4\,dz$$

$$= -6\frac{x^3}{3}\Big]_{-3}^{2} - 6\frac{y^2}{2}\Big]_{-3}^{6} - 4z\Big]_{2}^{-1}$$

$$= -2(8-(-27)) - 3(36-9) - 4(-1-2)$$

$$= -139\,V$$

(b) V_M if $V = 0$ at $Q(4, -2, -35)$;

$V_{MQ} = V_M - V_Q = V_M - 0 \Rightarrow V_M = V_{MQ}$

$$V_M = V_{MQ} = -\int_Q^M \vec{E}\cdot\vec{dl}$$

$$= -\int_4^2 6x^2\,dx - \int_{-2}^{6} 6y\,dy - \int_{-35}^{-1} 4z\,dz$$

$$= -6\frac{x^3}{3}\Big]_4^2 - 6\frac{y^2}{2}\Big]_{-2}^{6} - 4z\Big]_{-35}^{-1}$$

$$= -120\,V$$

(c) V_N if $V=2$ at $P(1,2,-4)$

$$V_{NP} = V_N - V_P \Rightarrow V_N = V_{NP} + V_P$$

$$V_{NP} = -\int_P^N \vec{E}\cdot\vec{dl}$$

$$= -\int_1^{-1} 6x^2 dx - \int_2^{-3} 6y\,dy - \int_{-4}^{2} 4\,dz$$

$$= -6\frac{x^3}{3}\Big]_1^{-3} - 6\frac{y^2}{2}\Big]_2^{-3} - 4z\Big]_{-4}^{2}$$

$$= -2(-27-1) - 3(9-4) - 4(2-(-4))$$

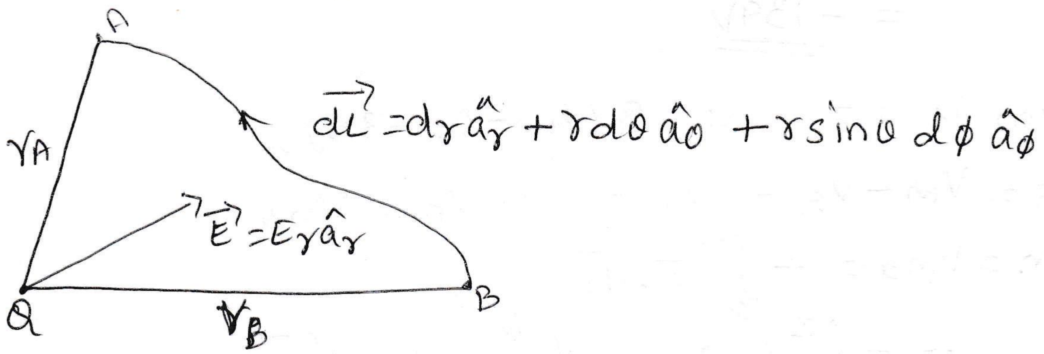
$$= \underline{17V}$$

$$\therefore V_N = N_{NP} + V_P = 17V + 2V = \underline{19V}$$

The Potential field of a point charge

The Potential difference b/w two points located at $r=r_A$ & $r=r_B$ in the field of a Point Charge Q placed at origin.

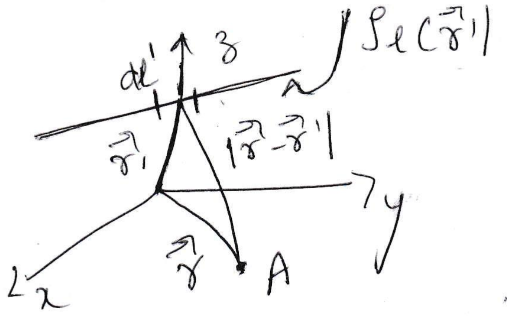
$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = V_A - V_B$$



If the point $r=r_B$ recede to ∞, the Potential at r_A is

$$\boxed{V_A = \frac{Q}{4\pi\varepsilon_0 r_A} = \frac{Q}{4\pi\varepsilon_0 r}}$$

If the charge distribution takes the form of a line Charge, then potential is given by



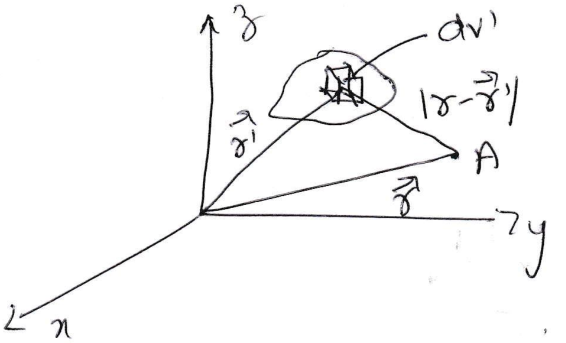
the differential Charge is given by

$$dQ = \rho_l(\vec{r}') \cdot dl'$$

$$\therefore dV_A = dV(\vec{r}) = \frac{dQ}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|}$$

$$\therefore V_A = V(\vec{r}) = \int \frac{\rho_l(\vec{r}') \cdot dl'}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|}$$

for Volume charge.

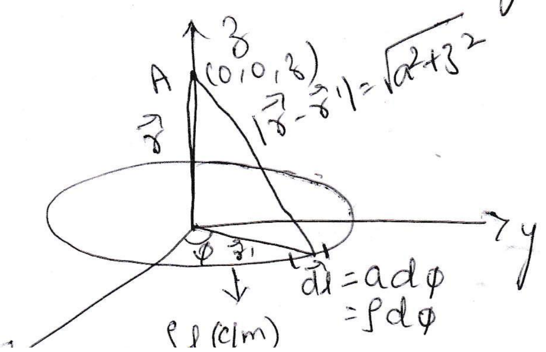


$$dQ = \rho_v(\vec{r}')\, dv'$$

$$\therefore dV_A = dV(\vec{r}) = \frac{dQ}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|}$$

$$\therefore V_A = V(\vec{r}) = \int_{vol} \frac{\rho_v(\vec{r}')\, dv'}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|}$$

Potential at a point on the Z-axis for a uniform line Charge 'ρ_l' in the form of a ring.



Let us Consider a uniform line charge having charge density ρ_l (C/m) in the form of a ring placed in $z=0$ plane

The differential length dl' at P on the ring is

$dl' = a\,d\phi$ (in cylindrical Co-ordinates) — ①

Differential charge is given by

$$dQ = \rho_l\, dl'$$

$$dQ = \rho_l\, a\, d\phi \quad — ②$$

the potential at point A due to dQ is,

$$dv = \frac{dQ}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|} \quad — ③$$

Using Eq (2) in Eq (3)

$$dv = \frac{\rho_l\, a\, d\phi}{4\pi\varepsilon_0\sqrt{a^2+z^2}} \qquad \because |\vec{r}-\vec{r}'| = \sqrt{a^2+z^2}$$

$\therefore$ the potential at point A due to entire line charge is

$$V = \int_{\phi=0}^{2\pi} \frac{\rho_l\, a\, d\phi}{4\pi\varepsilon_0\sqrt{a^2+z^2}}$$

$$V = \frac{\rho_l\, a}{4\pi\varepsilon_0\sqrt{a^2+z^2}} \left[\phi\right]_0^{2\pi}$$

$$\boxed{V = \frac{\rho_l\, a}{2\varepsilon_0\sqrt{a^2+z^2}}} \; V \quad — ④$$

$\rho_l \rightarrow$ line charge density (C/m)

$a \rightarrow$ radius of ring (m)

$\varepsilon_0 \rightarrow$ absolute potential (F/m).

An Equipotential Surface is a surface composed of all those points having the same value of Potential. No work is involved in moving a unit charge around on an equipotential surface.

1) A 15nc point charge is at origin in free space. Calculate V_1 if point P_1 is located at $P_1(-2,3,-1)$ and (a) $V_2 = 0$ at $P_2(6,5,4)$

given $V_2 = 0$

$\therefore V_{12} = V_1 - V_2$

$\therefore V_1 = V_{12} + V_2$

$\therefore V_{12} = -\int_{P_2}^{P_1} E_r\, dr = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$

$r_1 = \sqrt{3^2+2^2+1^2} = \underline{\underline{3.7}}$

$r_2 = \sqrt{6^2+5^2+4^2} = 8.77$

$= \frac{15\times10^{-9}}{4\pi\times8.854\times10^{-12}}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = 135\left[\frac{1}{3.74} - \frac{1}{8.77}\right] = \underline{20.7V}$

$\therefore V_1 = V_{12} = \underline{\underline{20.7V}}$

(b) $V_2 = 0$ at ∞

$V_{12} = V_1 - V_2$

$V_{12} = V_1 = \frac{Q}{4\pi\varepsilon_0 r_1} = \frac{15\times10^{-9}}{4\pi\times8.854\times10^{-12}\times\sqrt{14}} = \underline{\underline{36.09V}}$

(c) $V_2 = 5V$ at $(2,0,4)$

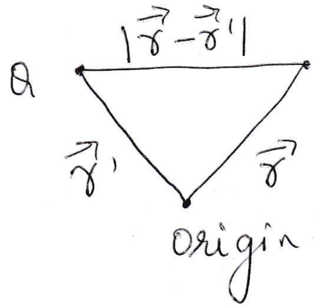
$V_{12} = V_1 - V_2$

$V_{12} = -\int_{P_2}^{P_1} E_r\, dr = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = 5.89V$

$V_1 = V_{12} + V_2 = 5.89 + 5 = \underline{\underline{10.89V}}$

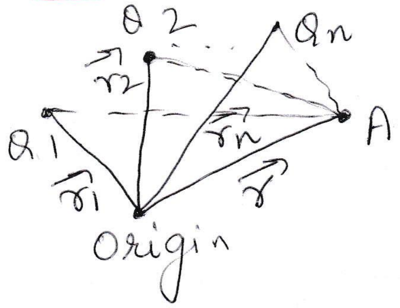
The potential field of a system of charges

The potential at a point A located at a distance $\vec{r}$ from the origin is



$$V_A = V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|}$$

Potential due to n-number of point charges.



the potential at point A located at a distance $\vec{r}$ from the origin due to Q_1 (at distance $\vec{r}_1$ from origin)

$$V_A = V_1(\vec{r}) = \frac{Q_1}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_1|}$$

III^{ly} the potential at point A due to Q_2

$$V_2(\vec{r}) = \frac{Q_2}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_2|}$$

$\therefore$ the potential at point A due to Qn is

$$V_n(\vec{r}) = \frac{Q_n}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_n|}$$

$\therefore$ the total potential arising from n point charges is

$$V(r) = \frac{Q_1}{4\pi\varepsilon_0|\vec{r}-\vec{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0|\vec{r}-\vec{r}_2|} - - - + \frac{Q_n}{4\pi\varepsilon_0|\vec{r}-\vec{r}_n|}$$

$$\boxed{V(r) = \sum_{m=1} \frac{Q_m}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_m|}}$$

Conservative field

The potential difference between two points A & B is given by (A is at higher potential).

$$V_{AB} = -\int_{B}^{A} \vec{E}\cdot\vec{dl} \quad \text{—— (1)}$$

Let a unit positive charge is taken through any arbitary path from point 'A' until it is brough back to point A. then

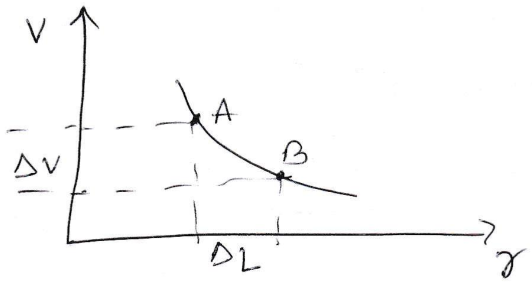
$$V = -\int_{A}^{A} \vec{E}\cdot\vec{dl} = 0$$

$$\boxed{\oint \vec{E}\cdot\vec{dl} = 0} \quad \text{— (2)}$$

no work is done in carrying the unit charge around any closed path.

Any field that satisfies an equation of the form of Eq (2) (where the closed line integral of the field is zero) is said to be a conservative field.

Potential Gradient



Consider $\vec{E}$ due to a Positive point charge at origin of a Sphere

$$V = -\int \vec{E} \cdot \vec{dl} = \frac{Q}{4\pi\varepsilon_0 r}$$

The potential decreases as the distance of Point from the charge increases. We know that the line integral of $\vec{E}$ between 2 points gives a potential difference between two Points. for an elementary length ΔL

we Can write

$$V_{AB} = \Delta V = -\vec{E} \cdot \vec{\Delta L}$$

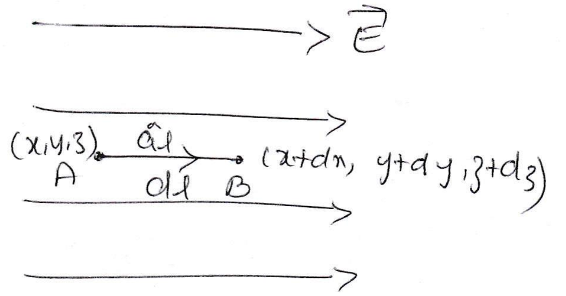
Hence an inverse relation namely the change of Potential ΔV along the elemental length ΔL must be related to $\vec{E}$ as $\Delta L \to 0$

∴ the rate of Change of Potential w.r.t distance is called the potential gradient.

$$\frac{dv}{dL} = \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \text{Potential gradient}$$

Relationship between $\vec{E}$ & V

Consider two neighbouring Points $A(x, y, z)$ and $B(x+dx, y+dy, z+dz)$ separated by a small distance dl in an electric field $\vec{E}$.



the work done in moving a charge Q from B to A is

$dw = -Q\vec{E}\cdot\vec{dl}$

By the definition of Potential,

if $Q=1$ then,

$$dv = dw$$

$$\therefore dv = -\vec{E}\cdot\vec{dl} \quad -(1)$$

In Cartesian Co-ordinates

$$\vec{E} = E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z \quad -(2)$$

& $$\vec{dl} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z \quad -(3)$$

using (2), (3) in Eq (1)

$$dv = -(E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z)\cdot(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z)$$

$$dv = -E_x dx - E_y dy - E_z dz \quad -(4)$$

We also know that potential is a function of (x,y,z).

$\therefore$ the change in potential is,

$$dv = dv_x + dv_y + dv_z$$

Change in potential along x, y, z-direction

$\therefore$ If rate of change of Potential are known then we can write

$$dv = \frac{\delta v}{\delta x}dx + \frac{\delta v}{\delta y}dy + \frac{\delta v}{\delta z}dz \quad -(5)$$

Comparing Eq (4) & Eq (5)

$$\frac{\delta v}{\delta x}dx + \frac{\delta v}{\delta y}dy + \frac{\delta v}{\delta z}dz = -\left[E_x dx + E_y dy + E_z dz\right]$$

Equating their Co-efficients (dx, dy, dz)

$$E_x = -\frac{\delta v}{\delta x}, \quad E_y = -\frac{\delta v}{\delta y}, \quad E_z = -\frac{\delta v}{\delta z} \quad \text{——} (6)$$

using Eq (6) in Eq (2)

$$\vec{E} = E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z \quad \text{—} (2)$$

$$\vec{E} = \left[-\frac{\delta v}{\delta x}\hat{a}_x - \frac{\delta v}{\delta y}\hat{a}_y - \frac{\delta v}{\delta z}\hat{a}_z\right]$$

$$\vec{E} = -\left(\frac{\delta}{\delta x}\hat{a}_x + \frac{\delta}{\delta y}\hat{a}_y + \frac{\delta}{\delta z}\hat{a}_z\right]v$$

$$\boxed{\vec{E} = -\nabla V}$$

$$\boxed{\vec{E} = -\text{gradient of } V}$$

V - Scalar

∇ - del operator → is a Vector ∴ gradient of a scalar is a vector.

thus, the electric field at any point is given by the negative of the gradient of potential at that point.

(1) $\nabla V = \frac{\delta v}{\delta x}\hat{a}_x + \frac{\delta v}{\delta y}\hat{a}_y + \frac{\delta v}{\delta z}\hat{a}_z$ (rectangular)

(2) $\nabla V = \frac{\delta v}{\delta \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\delta v}{\delta \phi}\hat{a}_\phi + \frac{\delta v}{\delta z}\hat{a}_z$ (cylindrical)

(3) $\nabla V = \frac{\delta v}{\delta r}\hat{a}_r + \frac{1}{r}\frac{\delta v}{\delta \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\delta v}{\delta \phi}\hat{a}_\phi$ (spherical)

1) If the potential field $V = 3x^2 + 3y^2 + 2z^3$ V find (1) V at P(-4, 5, 4) (2) $\vec{E}$ (3) $\vec{D}$ at P(-4,5,4)

(1) $V = 3x^2 + 3y^2 + 2z^3$

$= 3(-4)^2 + 3(5)^2 + 2(4)^3$

$V = 251\text{V}$

(2) $\vec{E} = -\nabla V = -\left[\frac{\delta V}{\delta x}\hat{a}_x + \frac{\delta V}{\delta y}\hat{a}_y + \frac{\delta V}{\delta z}\hat{a}_z\right]$

$= -\left[6x\hat{a}_x + 6y\hat{a}_y + 6z^2\hat{a}_z\right]$

$\vec{E} = 24\hat{a}_x - 30\hat{a}_y - 96\hat{a}_z$ V/m

(3) $\vec{D} = \varepsilon_0 \vec{E} = 0.215\hat{a}_x - 0.2656\hat{a}_y - 0.85\hat{a}_z$ nC/m²

(2) Find the electric field strength at the point M(1, 2, -1) given the potential $V = 3x^2y + 2yz^2 + 3xyz$

$\vec{E} = -\nabla V$

$\vec{E} = -\left[\frac{\delta V}{\delta x}\hat{a}_x + \frac{\delta V}{\delta y}\hat{a}_y + \frac{\delta V}{\delta z}\hat{a}_z\right]$

$\vec{E} = -\left[(6xy + 3yz)\hat{a}_x + (3x^2 + 2z^2 + 3xz)\hat{a}_y + (4yz + 3xy)\hat{a}_z\right]$

at P(1, 2, -1)

$\vec{E} = -6\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z$ V/m

(3) given potential field $V = 2x^2y - 5z$ and a point $P(-4, 3, 6)$. Find $V, \vec{E}$ & direction of $\vec{E}$, D at P and ρ_{vp} at P.

Soln:- (1) $V_P = 2x^2y - 5z$

$$= 2(-4)^2(3) - 5(6) = 2\times16\times3 - 30$$

$$V_P = 96 - 30 = \underline{66V}$$

(2) $\vec{E} = -\nabla V \Rightarrow \left(\frac{\delta V}{\delta x}\hat{a}_x + \frac{\delta V}{\delta y}\hat{a}_y + \frac{\delta V}{\delta z}\hat{a}_z\right)$

$$\vec{E} = -4xy\hat{a}_x - 2x^2\hat{a}_y + 5\hat{a}_z \quad - (1)$$

$$\vec{E}_P = -4(-4)(3)\hat{a}_x - 2(-4)^2\hat{a}_y + 5\hat{a}_z \text{ V/m}$$

$$\vec{E}_P = 48\hat{a}_x - 32\hat{a}_y + 5\hat{a}_z \text{ V/m}$$

$|\vec{E}_P| = \underline{57.9 \text{ V/m}}$

(3) direction of $\vec{E}$ at P $\Rightarrow \hat{a}_{E_P}$

$$= \frac{48\hat{a}_x - 32\hat{a}_y + 5\hat{a}_z}{57.9}$$

$$= 0.829\hat{a}_x - 0.553\hat{a}_y + 0.086\hat{a}_z$$

(4) $\vec{D}_P = \varepsilon_0\vec{E}_P = 8.854\times10^{-12}\left[48\hat{a}_x - 32\hat{a}_y + 5\hat{a}_z\right]$

$$= 0.425\hat{a}_x - 0.2833\hat{a}_y + 0.0443\hat{a}_z \text{ nC/m}^2$$

(5) $\rho_v = \nabla\cdot\vec{D} = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$ $\qquad \vec{D} = \varepsilon_0\cdot\vec{E}$

using Eq (1)

$$\rho_v = \varepsilon_0\left[-4y + 0 + 0\right] = -\underline{\varepsilon_0 4y} \text{ C/m}^3$$

$$\rho_v \text{ at } P = -4(3)\varepsilon_0 = \underline{-0.1062 \text{ nC/m}^3}$$

(4) An electric potential is given by

$V = \frac{60 \sin\theta}{r^2}$ V find V & $\vec{E}$ at $P(\underset{r}{3}, \underset{\theta}{60°}, \underset{\phi}{25°})$

(1) $V = \frac{60 \sin 60°}{(3)^2} = \underline{\underline{5.7735\, V}}$ (degree mode)

(2) $\vec{E} = -\nabla V$

$$= -\left(\frac{\delta V}{\delta r} \hat{a}_r + \frac{1}{r} \frac{\delta V}{\delta \theta} \hat{a}_\theta + \frac{1}{r \sin\theta} \frac{\delta V}{\delta \phi} \hat{a}_\phi \right)$$

$$\vec{E} = -\left[-\frac{120 \sin\theta}{r^3} \hat{a}_r + \frac{1}{r} \frac{60 \cos\theta}{r^2} \hat{a}_\theta \right]$$

$$\vec{E}_p = -\left[-\frac{120 \sin 60°}{(3)^2} \hat{a}_r + \frac{60 \cos 60°}{(3)^3} \hat{a}_\theta \right]$$

$$\vec{E}_p = \underline{\underline{3.849 \hat{a}_r - 1.111 \hat{a}_\theta \text{ V/m}}}$$

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(5) Given $V = \frac{100 \rho \cos\phi}{z^2+1}$ V & $P(3, 60°, 2)$ find at P (a) V (b) $\vec{E}$ (c) $|\vec{E}|$ (d) $\hat{a}_N$ (e) ρ_v in free space.

(a) $V_p = \frac{100}{(2)^2+1} \times 3 \times \cos 60° = \underline{\underline{30 V}}$

(b) $\vec{E} = -\nabla V = -\left(\frac{\delta V}{\delta \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\delta V}{\delta \phi} \hat{a}_\phi + \frac{\delta V}{\delta z} \hat{a}_z \right)$

$$= -\left[\frac{100}{z^2+1} \cos\phi \, \hat{a}_\rho + \frac{1}{\rho} \frac{100}{z^2+1} (-\rho \sin\phi) \hat{a}_\phi + \left(\frac{-2z}{(z^2+1)^2} \right) 100 \rho \cos\phi \, \hat{a}_z \right]$$

$$\vec{E}_p = -10 \hat{a}_\rho + \underline{\underline{17.32 \hat{a}_\phi}} + 24 \hat{a}_z \text{ V/m}$$

$|\vec{E}_p| = \underline{\underline{31.2\ V/m}}$

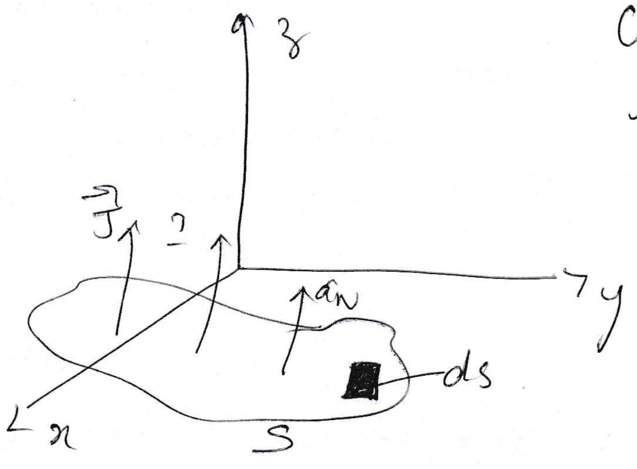
Current & Current density

Flow of charge per unit time i.e, rate of flow of charge at a specified point or across a specified surface is called an electric current.

Mathematically $I = \frac{dQ}{dt}$ c/sec i.e, Amps.

Current density $\vec{J}$ is a vector defined as the current passing through the unit surface area when the surface is held normal to the direction of current A/m^2.

Relation between $\vec{J}$ & I.



Consider a surface S & I is the current passing through the surface. The direction of I is normal to the surface normal to the surface $\therefore \vec{J}$ is normal to S.

The incremental ΔI crossing an incremental surface ΔS normal to the current density is

$$\Delta I = J_N \Delta S$$

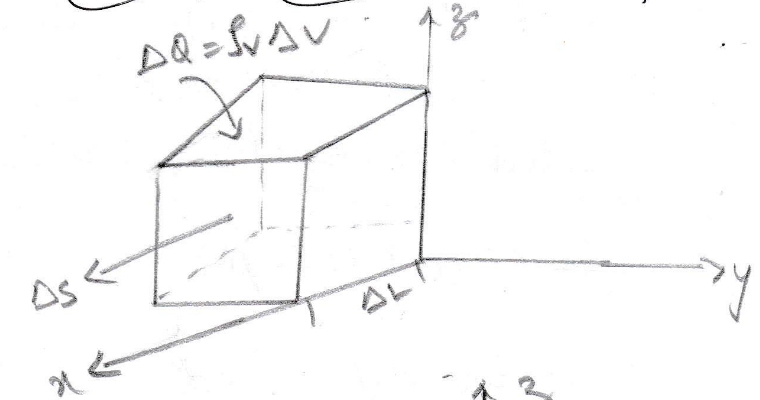
$$\Delta I = \vec{J} \cdot \vec{\Delta S}$$

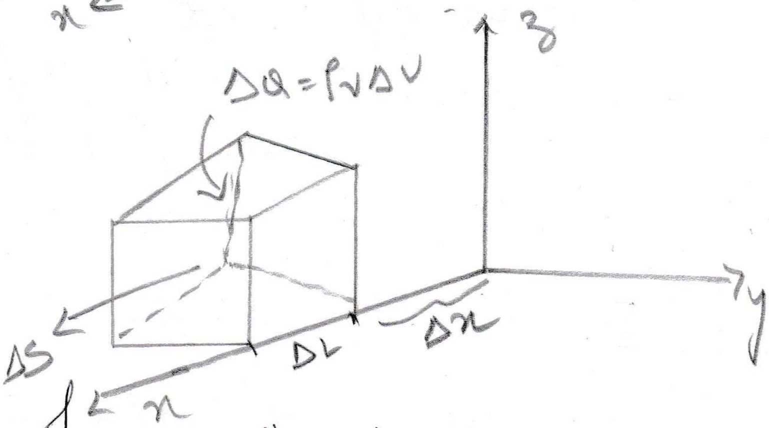
$\vec{ds} = ds \cdot \hat{a}_n$

$\vec{J} = J \hat{a}_n$

$\therefore$ total current $I = \int_S \vec{J} \cdot \vec{ds}$ A

Relation b/w $\vec{J}$ and ρ_v





Consider a differential volume ΔV having ρ_v charge density.
The elementary charge this volume carries is
$$\Delta Q = \rho_v \Delta V$$
ΔL = incremental length
ΔS = incremental surface area
hence incremental volume is $\Delta V = \Delta S \Delta L$
$$\therefore \Delta Q = \rho_v \Delta S \Delta L$$

In the time interval Δt the element of charge has moved through distance Δx in x-direction the direction is normal to the surface ΔS & hence resultant current can be expressed as

$$\Delta I = \frac{\Delta Q}{\Delta t}.$$

But $\Delta Q = \rho_s \Delta S \Delta x$ → as the charge corresponding the length Δx is moved & responsible for the current.

$$\Delta I = \rho_v \Delta S \frac{\Delta x}{\Delta t}$$

$$\Delta I = \rho_v \Delta S V_x$$ → x-component of velocity $\vec{v}$

$$\vec{J}_x = \rho_v \vec{V}_x$$

$$\boxed{\vec{J} = \rho_v \vec{V}}$$

This indicates charge in motion constitutes a convection current
$\vec{J}$ = Convection current density

(1) Given $\vec{J} = 10\rho^2 z\, \hat{a}_\rho - 4\rho \cos^2\phi\, \hat{a}_\phi$ mA/m^2

(a) find the current density at $P(3, 30°, 2)$

(b) determine the total current flowing outward through the circular band $\rho = 3$, $0 < \phi < 2\pi$, $2 < z < 2.8$.

(a) $\vec{J}_P = 10 \times (3)^2 \times (2)\, \hat{a}_\rho - 4 \times 3 \times \cos^2(30°)\, \hat{a}_\phi$

$= 180\hat{a}_\rho - 9\hat{a}_\phi$ mA/m^2

(b) $$I = \int_S \vec{J} \cdot \vec{ds} = \int_{z=2}^{2.8} \int_{\phi=0}^{2\pi} (10\rho^2 z)\, \hat{a}_\rho\, \rho\, d\phi\, dz\, \hat{a}_\rho$$

$$= \int_{z=2}^{2.8} z\, dz \int_{\phi=0}^{2\pi} 10\rho^3 d\phi = \frac{z^2}{2}\Big]_2^{2.8} (2\pi)(10)(3)^3$$

$= 3.257A$

Continuity Equation

The Continuity equation of the current is based on "the principle of Conservation of Charges, it states that " Charges can neither be created nor be destroyed".

Let us Consider any region bounded by a closed surface, the current through the closed surface is

$$I = \oint_S \vec{J} \cdot \vec{ds} \quad \text{--- (1)}$$

the above equation refers to the net outward flow of charges from the closed surface. This should be balanced by a decrease of charge by the same amount within the closed surface.

If the charge inside the closed surface is denoted by Q_i, then the rate of decrease of charge is

$\overline{dt}$

$\therefore$ law of Conservation of charges

$$I = \oint_s \vec{J} \cdot \vec{ds} = -\frac{dQ_i}{dt} \quad -(2)$$

W.K.T the divergence theorem is,

$$\oint_s \vec{J} \cdot \vec{ds} = \int_{vol} (\nabla \cdot \vec{J}) dv \quad -(3)$$

Equating eq (2) & (3)

$$\int_{Vol} (\nabla \cdot \vec{J}) dv = -\frac{dQ_i}{dt}$$

But $Q_i = \int_{vol} \rho_v dv$

$$\therefore \int_{Vol} (\nabla \cdot \vec{J}) dv = -\frac{d}{dt}\left[\int_{Vol} \rho_v dv\right]$$

When we take $\frac{d}{dt}$ inside the integral, it becomes ρ_v ~~[illegible]~~ is also a function of spatial Co-ordinates

then $$\int_{Vol} (\nabla \cdot \vec{J}) dv = -\int_{vol} \frac{\delta \rho_v}{\delta t} dv \quad -(4)$$

the above equation should be true for any volume however small it is, then we have

$$\nabla \cdot \vec{J} \Delta v = -\frac{\delta \rho_v \Delta v}{\delta t}$$

$$\boxed{\nabla \cdot \vec{J} = -\frac{\delta \rho_v}{\delta t}} \quad -(5)$$ differential / Point form of Continuity equation.

States that " the current or ~~[illegible]~~ Charge/sec diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

1) The current density is given in cylindrical co-ordinates as $\vec{J} = -10^6 z^{1.5} \hat{a}_z$ A/m2 in the region $0 \leq \rho \leq 20\mu m$; for $\rho \geq 20\mu m$, $\vec{J} = 0$

(a) find the total current crossing the surface $z = 0.1m$ in the $\hat{a}_z$ direction.

(b) If the velocity is $2\times10^6 m/s$ at $z = 0.1m$. find ρ_v

(c) If the volume charge density at $z = 0.15m$ is $-2000C/m^3$ find the charge velocity there

Given:- $\vec{J} = -10^6 z^{1.5} \hat{a}_z \quad 0 \leq \rho \leq 20\mu m$

$= 0 \quad \rho \geq 20\mu m$

$$I = \int \vec{J} \cdot \vec{ds} = \int_0^{20\mu} \int_0^{2\pi} (-10^6 z^{1.5} \hat{a}_z) \cdot (\rho \, d\phi \, d\rho) \hat{a}_z$$

$$= -10^6 z^{1.5} \int_0^{20\mu} \rho \, d\rho \int_0^{2\pi} d\phi$$

$$= -10^6 [0.1]^{1.5} \times \left. \frac{\rho^2}{2} \right]_0^{20\mu} \times 2\pi$$

$$= \frac{-10^6 (0.1)^{1.5}}{2} \times (20\times10^{-6})^2 (2\pi)$$

$$= 39.74 \mu A$$

W.K.T $\vec{J} = \rho_v \vec{V} \Rightarrow \vec{J} = \rho_v |\vec{V}|$

$$\rho_v = \frac{J}{|\vec{V}|} = \frac{-10^6 z^{1.5}}{2\times10^6} = \frac{-10^6 (0.1)^{1.5}}{2\times10^6}$$

$\rho_v = -15.8 \text{ mC/m}^3$