5th sem EC EMW

70

Module - 4

Magnetic Forces & Magnetic Materials

Topics Covered

Force on a moving charge, differential current elements, Force b/w differential current elements.

Magnetisation & permeability.

Magnetic boundary Conditions

Magnetic circuit

Potential energy & force on magnetic matirials

Books Referred

1. Engineering Electromagnetics - W.H. Hayt & J.A BUCK, Mc-Graw Hill Education (India) Private Limited, 7th Edition.

2. "principles of Electromagnetics" - Matthew N.O. Sadiku - Oxford International student Edition - 4th Edition -

Integrating Eq (4) over open or closed surface is

$$\vec{F} = \int_S \vec{K} \times \vec{B} \, ds .$$

IIIly integrating Eq (5) over a closed path we get (ie, entire length)

$$\boxed{\vec{F} = \oint_l I \, d\vec{L} \times \vec{B} .} \; N$$

Introduction

Electric field exerts a force on either stationary or moving charge while the steady magnetic field can exert a force on only moving charges.

Force on a moving point charge

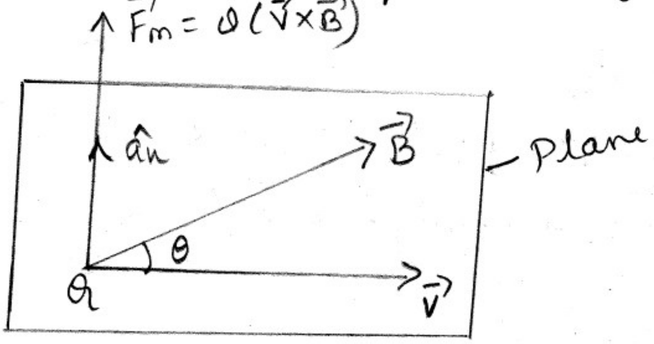
If a charge of Q Coulumbs is present in an electric field of strength $\vec{E}$ then the force experienced by the charge Q is

$$\boxed{\vec{F_e} = Q\vec{E}} \quad N$$

Consider a charge which is placed in a steady magnetic field of strength '$\vec{B}$' Wb/m²
then the magnetic force ($\vec{F_m}$) exerted on a charge Q moving with a velocity $\vec{V}$ in a steady magnetic field $\vec{B}$ is given by

$$\boxed{\vec{F_m} = Q\,[\vec{V} \times \vec{B}]} \quad N$$

[Note:- the direction of force is perpendicular to the plane containing $\vec{V}$ & $\vec{B}$.
the magnitude of the magnetic force $\vec{F_m}$ is proportional to the magnitude of charge Q, $\vec{V}$ & $\vec{B}$ and also the sine of the angle between $\vec{V}$ & $\vec{B}$].



Lorentz Force Equation

If a charge of 'Q' Coulumb is moving with a velocity of 'V' m/sec in a Combined electric & magnetic fields ~~then~~ the total force acting on ~~the~~ charge is

$$\vec{F} = \vec{F_e} + \vec{F_m} = Q\vec{E} + Q(\vec{V} \times \vec{B})$$

$$\boxed{\vec{F} = Q(\vec{E} + \vec{V} \times \vec{B}) \; N}$$

(1) term $Q\vec{E}$ does not depend on $\vec{V}$ & $\vec{F_e}$ is in the same direction as $\vec{E}$.

(2) $Q(\vec{V} \times \vec{B})$ depends on $\vec{V}$ & perpendicular to both $\vec{V}$ & $\vec{B}$.

1) A point charge of $Q = -1.2c$ has velocity $\vec{V} = (5\hat{a}_x + 2\hat{a}_y - 3\hat{a}_z)$. Find the magnitude of the force exerted on the charge if (i) $\vec{E} = -18\hat{a}_x + 5\hat{a}_y - 15\hat{a}_z$ (ii) $\vec{B} = -4\hat{a}_x + 4\hat{a}_y + 3\hat{a}_z$ T (iii) Both are present simultaneously.

Soln:- (i) Electric force exerted by $\vec{E}$ on charge Q is

$$\vec{F_e} = Q\vec{E} = -1.2[-18\hat{a}_x + 5\hat{a}_y - 10\hat{a}_z]$$

$$= 21.6\hat{a}_x - 6\hat{a}_y + 12\hat{a}_z \; N$$

$$|\vec{F_e}| = \sqrt{(21.6)^2 + (-6)^2 + (12)^2} = \underline{25.427} \; N$$

(ii) $\vec{F_m} = Q(\vec{V} \times \vec{B})$

$$= -1.2[(5\hat{a}_x + 2\hat{a}_y - 3\hat{a}_z) \times (-4\hat{a}_x + 4\hat{a}_y + 3\hat{a}_z)]$$

$$= -21.6\hat{a}_x + 3.6\hat{a}_y - 33.6\hat{a}_z \; N$$

$$|\vec{F_m}| = \underline{\underline{40.105}} \; N$$

(iii) $\vec{F} = \vec{F_e} + \vec{F_m} = Q\left(\vec{E} + (\vec{v} \times \vec{B})\right)$

$$= \left[(21.6\hat{a}_x - 6\hat{a}_y + 12\hat{a}_z) + (-21.6\hat{a}_x + 3.6\hat{a}_y - 33.6\hat{a}_z)\right)$$

$$= 0\hat{a}_x - 2.4\hat{a}_y - 21.6\hat{a}_z \text{ N}$$

$$|\vec{F}| = 21.73 \text{ N}$$

Force on a differential current Element

When a current is flowing in a conductor, electrons are in motion within the conductor. If the current carrying conductor is present within a magnetic field of flux density $\vec{B}$ Wb/m2 each and every electron within the conductor experiences a force given by

$$\vec{dF} = dQ\,(\vec{V} \times \vec{B}) \quad -①$$

We have defined convection current density in terms of the velocity of the volume charge density

$$\vec{J} = \rho_v \vec{V} \quad -②$$

The differential element of charge may also be expressed in terms of volume charge density

$$dQ = \rho_v\,dv$$

thus, $\vec{dF} = \rho_v\,dv\ \vec{V} \times \vec{B}$ $\quad [\vec{J} = \rho_v \vec{V}]$

$$\boxed{\vec{dF} = \vec{J} \times \vec{B}\,dv} \quad -①$$

$\vec{J}\,dv$ may be interpreted as a differential current element i.e,

$$\vec{J}dv = \vec{K}ds = I\vec{dL}$$

thus the Lorentz force equation may be applied to surface current density

$$\boxed{\vec{dF} = \vec{K} \times \vec{B}\, ds} \quad -(2)$$

or to a differential current filament

$$\boxed{\vec{dF} = I\,\vec{dL} \times \vec{B}} \quad — (3)$$

Integrating over a volume (Eq (1), (2), (3)) or over a closed surface leads to

$$\vec{F} = \int_{vol} \vec{J} \times \vec{B}\, dv$$

$$\vec{F} = \oint_{s} \vec{K} \times \vec{B}\, ds$$

$$\therefore \boxed{\vec{F} = \oint_{l} I\,\vec{dL} \times \vec{B}} \quad \text{or} \quad \boxed{\vec{F} = -I \oint_{l} \vec{B} \times \vec{dL}} \quad (4)$$

if we apply eq (4) to a st conductor in a uniform magnetic field,

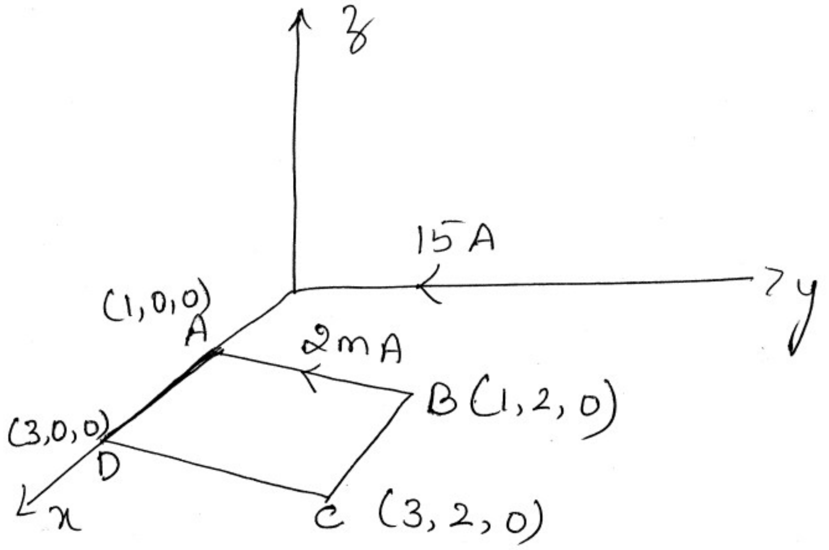
$$\therefore \boxed{\vec{F} = I\vec{L} \times \vec{B}}$$

the magnitude of the force is given by

$$\boxed{\vec{F} = BIL\sin\theta}$$

where θ is the angle between the vectors representing the direction of current flow & the direction of the magnetic flux density.

1. A square loop carrying 2mA current is placed in the field of an infinite filament carrying current of 15A as shown. Find the force exerted on the loop.



Soln:- The force exerted on the loop is given by

$$\vec{F} = \oint I \, d\vec{L} \times \vec{B}$$

$$\vec{F} = -I \oint \vec{B} \times d\vec{L}$$

the current is along -ve direction of y-axis, the magnetic field produced at a point in it which is at a normal distance x from the ∞ current element is

$$\vec{H} = \frac{I}{2\pi x} \hat{a}_z \ A/m \qquad I = 15A$$

$$\vec{B} = \mu_0 \vec{H} = 4\pi \times 10^{-7} \left[\frac{15}{2\pi x}\right] \hat{a}_z$$

$$\vec{B} = \frac{30 \times 10^{-7}}{x} \hat{a}_z$$

∴ the total force $\vec{F}$ is the sum of the forces on each of the 4 sides of the loop.

Let us begin with the side co-inciding with the x-axis & go anticlockwise,

$x = 1$ to 3, $y = 0$ to 2, $x = 3$ to 1, $y = 2$ to 0.

$I = 2\times10^{-3}A$ in the loop.

$$\vec{F} = -2\times10^{-3}\left[\int_{x=1}^{3} \frac{30\times10^{-7}}{x}\hat{a}_z \times dx\,\hat{a}_x + \int_{y=0}^{2} \frac{30\times10^{-7}}{3}\hat{a}_z \times dy\,\hat{a}_y \right.$$

at $x=3$

$$\left. + \int_{x=3}^{1} \frac{30\times10^{-7}}{x}\hat{a}_z \times dx\,\hat{a}_x + \int_{y=2}^{0} \frac{30\times10^{-7}\hat{a}_z}{1} \times dy\,\hat{a}_y\right]$$

$$\vec{F} = -6\times10^{-9}\left[(\ln x)_1^3\,\hat{a}_y + \frac{(y)_0^2}{3}(-\hat{a}_x) + \ln(x)_1^3\,\hat{a}_y + (y)_0^2(-\hat{a}_x)\right]$$

$$= -6\times10^{-9}\left[(\ln 3 - \ln 1)\hat{a}_y + \left(\frac{2-0}{3}\right)(-\hat{a}_x) + (\ln 1 - \ln 3)\hat{a}_y + (0-2)(-\hat{a}_x)\right]$$

$$= -6\times10^{-9}\left[(1.0986 - 0)\hat{a}_y - \frac{2}{3}\hat{a}_x + (0 - 1.0986)\hat{a}_y + 2\hat{a}_x\right]$$

$$= -6\times10^{-9}(1.333\,\hat{a}_x)$$

$$= \underline{-8\hat{a}_x\,nN}$$

(force on the loop is in the $-\hat{a}_z$ direction).

(2) The field $\vec{B} = -2\hat{a_x} + 3\hat{a_y} + 4\hat{a_z}$ mT is present in free space. Find the vector force exerted on a straight wire carrying 12 A in $\hat{a}_{AB}$ direction given A (1, 1, 1) & (i) B (3, 5, 6) (ii) B(2,1,1)

Soln:-

(i) W.K.T, $\vec{F} = -I\int_A^B \vec{B}\times d\vec{L}$

$\therefore \vec{F} = -I\,\vec{B}\times\int_A^B d\vec{L}$

$= -I\vec{B}\times\vec{L}_{AB}$

$\vec{F} = -I\vec{B}\times\vec{R}_{AB}$

$$\vec{F} = -12\begin{vmatrix}\hat{a_x} & \hat{a_y} & \hat{a_z}\\ -2 & 3 & 4\\ 2 & 4 & 5\end{vmatrix}$$

$= -12\left[\hat{a_x}(-1) - \hat{a_y}(-18) + \hat{a_z}(-14)\right]$ mN

$\vec{F} = -12\left[-\hat{a_x} + 18\hat{a_y} - 14\hat{a_z}\right]$ mN

$= 12\hat{a_x} - 216\hat{a_y} + 168\hat{a_z}$ mN

(ii) $\vec{F} = -I\int_A^B \vec{B}\times d\vec{L}$

$= -I\,\vec{B}\times\int_A^B d\vec{L}$

$= -I\,\vec{B}\times\vec{L}_{AB}$

$\vec{L}_{AB} = (2-1)\hat{a_x} = \hat{a_x}$

$= -12\left(-2\hat{a_x} + 3\hat{a_y} + 4\hat{a_z}\right)\times(\hat{a_x})$

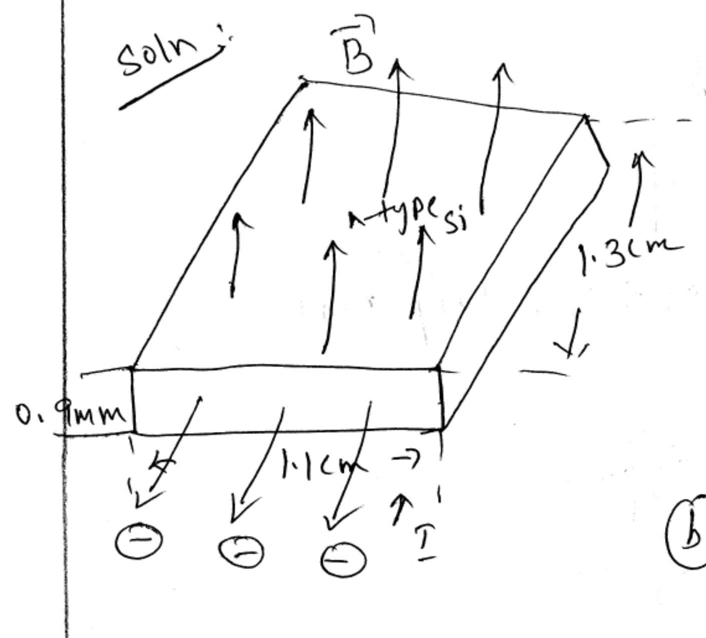
$= \left(24\hat{a_x} - 36\hat{a_y} - 48\hat{a_z}\right)\times\hat{a_x}$

$= -36(-\hat{a_z}) - 48\hat{a_y}$

$= -48\hat{a_y} + 36\hat{a_z}$ mN

(3) A Semiconductor Sample shown below is n-type silicon, having a rectangular cross-section of 0.9mm by 1.1cm & a length of 1.3 cm. Assume the electron & hole mobilities are 0.13 & 0.03 m^2/Vs, respectively at the operating temperature. Let B = 0.07T & the electric field intensity in the direction of the current flow be 800V/m. Find the magnitude of
(a) the voltage across the sample length.
(b) the drift velocity
(c) the transverse force per coulomb of moving charge caused by B.
(d) the transverse electric field intensity (e) Hall Voltage

soln :



(a) Voltage across the sample length = Field in the direction of current flow × length of sample

$= 800 \times 1.3 \times 10^{-2}$ (E×L)

$= \underline{10.4V}$

(b) Vd = (mobility)(field intensity)

$= 0.13 \times 800 = 104 m/sec$

(c) transverse force/coulomb $= \frac{F}{Q} = V_d B = 104(0.07)$

$= \underline{7.28\ N/c}$

(d) transverse electric field $= \frac{F}{Q} = V_d B = 7.28V/m$.

(e) hall voltage = E (d)

$= 7.28 \times 1.1 \times 10^{-2}$

$= \underline{80.1mV}$

Numericals:-

① The Point charge, $Q = 18nc$ has a velocity of 5×10^6 m/sec in the direction $\hat{a}_v = 0.6\hat{a}_x + 0.75\hat{a}_y + 0.3\hat{a}_z$ Calculate the magnitude of force exerted on the charge by the field ⓐ $\vec{B} = -3\hat{a}_x + 4\hat{a}_y + 6\hat{a}_z$ mT.

ⓑ $\vec{E} = -3\hat{a}_x + 4\hat{a}_y + 6\hat{a}_z$ kV/m ⓒ B & E acting together

ⓐ Force exerted on Q by $\vec{B}$.

$$\vec{F} = Q(\vec{V} \times \vec{B}) = QV(\vec{a}_V \times \vec{B})$$

$$= 18 \times 10^{-9} \times 5 \times 10^{6} \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0.6 & 0.75 & 0.3 \\ -3 & 4 & 6 \end{vmatrix}$$

$$\vec{F} = 90 \times 10^{-3} \left[\hat{a}_x(4.5 - 1.2) - \hat{a}_y(3.6 + 0.9) + \hat{a}_z(2.4 + 2.25)\right] mN$$

$$\vec{F} = 90 \times 10^{-6} \left[3.3\hat{a}_x - 4.5\hat{a}_y + 4.65\hat{a}_z\right] N$$

$$\vec{F} = (297\hat{a}_x - 405\hat{a}_y + 418.5\hat{a}_z) \times 10^{-6} N$$

$$|\vec{F}| = 653.74 \ \mu N$$

ⓑ $\vec{E} = -3\hat{a}_x + 4\hat{a}_y + 6\hat{a}_z$ kV/m.

$$\vec{F} = Q\vec{E} = 18 \times 10^{-9} \left[-3\hat{a}_x + 4\hat{a}_y + 6\hat{a}_z\right] \times 10^{3} N$$

$$= 18 \times 10^{-6} (-3\hat{a}_x + 4\hat{a}_y + 6\hat{a}_z) N$$

$$= -54\hat{a}_x + 72\hat{a}_y + 108\hat{a}_z \ \mu N$$

$$\therefore |\vec{F}| = 140.58 \ \mu N$$

(c) Force due to both $\vec{E}$ & $\vec{B}$.

$$\vec{F} = (243\,\hat{a}_x - 333\,\hat{a}_y + 526.5\,\hat{a}_z) \times 10^{-6}$$

$$|\vec{F}| = 668\ \mu N$$

P.T.O

Force between differential current elements

The magnetic field at point 2 due to a current element at point 1 was found to be

$$\vec{dH}_2 = \frac{I_1 \, \vec{dL}_1 \times \hat{a}_{R12}}{4\pi R_{12}^2}$$

the differential force on a differential current element is $\vec{dF} = I \, \vec{dL} \times \vec{B}$

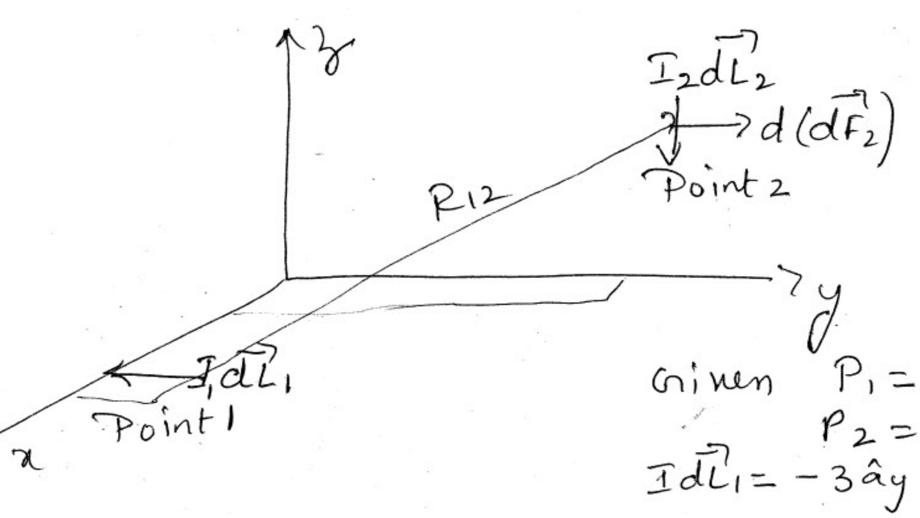
By allowing $\vec{B}$ to be $\vec{dB}_2$ (the differential flux density at point 2 caused by current element 1) by identifying $I\vec{dL}$ as $I_2 \vec{dL}_2$ & by symbolizing the differential amount of differential force on element 2 as $d(\vec{dF}_2)$

$$d(\vec{dF})_2 = I_2 \, \vec{dL}_2 \times \vec{dB}_2$$

Since $\vec{dB}_2 = \mu_0 \vec{dH}_2$, $\therefore$ the force b/w two differential current elements,

$$\boxed{d(\vec{dF}_2) = \mu_0 \frac{I_1 I_2}{4\pi R_{12}^2} \vec{dL}_2 \times (dL_1 \times \hat{a}_{R12})}$$

1) find the differential force on $\vec{dL}_2$



Given $P_1 = (5, 2, 1)$
$P_2 = (1, 8, 5)$
$I\vec{dL}_1 = -3\hat{a}_y$ A.m, $I\vec{dL}_2 = -4\hat{a}_z$ A.m

Soln :- $I_1 d\vec{L_1} = -3\hat{a}_y$ A·m at $P_1 (5,2,1)$

$I_2 d\vec{L_2} = -4\hat{a}_z$ A·m at $P_2 (1,8,5)$

thus $\vec{R}_{12} = (1-5)\hat{a}_x + (8-2)\hat{a}_y + (5-1)\hat{a}_z$

$= -4\hat{a}_x + 6\hat{a}_y + 4\hat{a}_z$

$$d(d\vec{F}_2) = \frac{\mu_0 I_1 I_2 \, d\vec{L}_2 \times (d\vec{L}_1 \times \hat{a}_{R_{12}})}{4\pi R_{12}^2}$$

$$= \frac{4\pi 10^{-7}}{4\pi} \frac{(-4\hat{a}_z) \times \left[(-3\hat{a}_y) \times (-4\hat{a}_x + 6\hat{a}_y + 4\hat{a}_z)\right]}{(16+36+16)^{1.5}}$$

$$= \underline{\underline{8.56\,\hat{a}_y \text{ nN}}}$$

Magnetic Torque

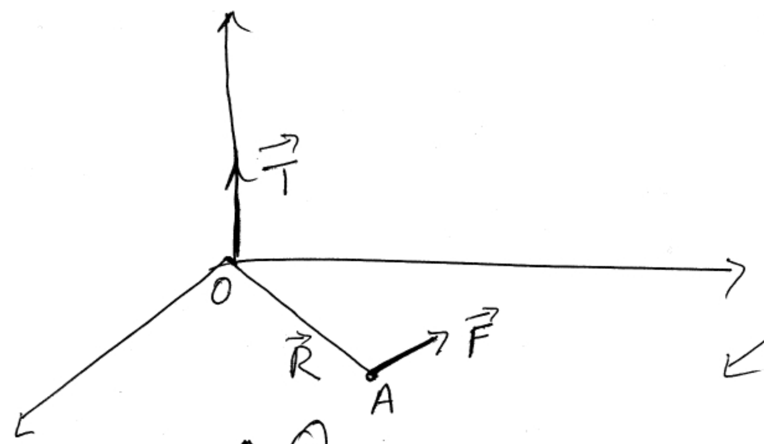


fig (a)

Torque $\vec{T}$ about the origin

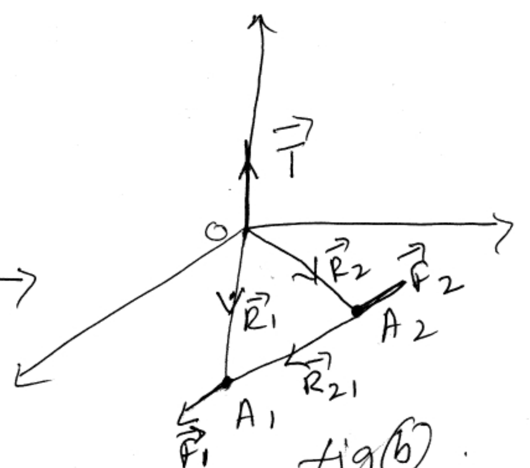


fig (b)

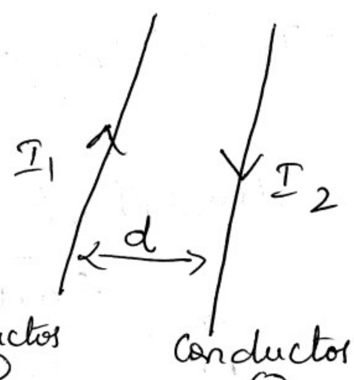
Torque about the origin with $\vec{F}_2 = -\vec{F}_1$

[Continued on page 9]

Force between two parallel Conductors

Consider two infinite parallel Conductors having the Separation 'd' & carrying current 'I' in opposite direction. The magnetic field due to Conductor 1 on the Conductor 2 is uniform & is given by

$$\vec{B}_2 = \frac{\mu_0 I_1}{2\pi d} \hat{a}_N$$



Where $\hat{a}_N \Rightarrow$ is the unit vector normal to plane Consisting of two Conductors & is in the advancement of RH Screw when it is turned from Conductor (1) to Conductor (2).

W.K.T the force on current carrying Conductor lying in uniform magnetic field is given by,

$$\vec{F} = I \vec{L} \times \vec{B}.$$

$\therefore$ we can write

$$\vec{F}_2 = I_2 \vec{L} \times \vec{B}_2$$

$$\vec{F}_2 = I_2 L \hat{a}_L \times \left(\frac{I_1 \mu_0}{2\pi d}\right) \hat{a}_N$$

[Where $\hat{a}_L$ & $\hat{a}_N$ are $\perp r$ to each other]

$$\vec{F}_2 = \frac{\mu_0 I_1 I_2 L}{2\pi d} (\hat{a}_L \times \hat{a}_N)$$

$$\vec{F}_2 = \frac{\mu_0 I_1 I_2 L}{2\pi d} \hat{a}_d$$

[Where $\hat{a}_d$ is the unit vector in the increasing direction of radial distance b/w two Conductors]

$$\frac{\vec{F}_2}{L} = \frac{\mu_0 I_1 I_2}{2\pi d} \hat{a}_d$$

$$\frac{F_2}{L} = \frac{\mu_0 I_1 I_2}{2\pi d}$$ force per unit length of Conductor 2 due to Conductor 1

$\text{ully},\quad \vec{F_1} = I_1 \vec{L} \times \vec{B_1}$

where, $\vec{B_1} = \frac{\mu_0 I_2}{2\pi d} \hat{a}_N$

$$\vec{F_1} = I_1 \left[L(-\hat{a}_L)\right] \times \frac{\mu_0 I_2 \hat{a}_N}{2\pi d}$$

$$\vec{F_1} = \frac{\mu_0 I_1 I_2 (-\hat{a}_d)}{2\pi d}$$

$$\therefore \frac{\vec{F_1}}{L} = \frac{\mu_0 I_1 I_2}{2\pi d}(-\hat{a}_d)$$

$$\therefore \frac{\vec{F_1}}{L} = \frac{\mu_0 I_1 I_2}{2\pi d} \text{ N/m}$$ force per unit length on conductor 1 due to Conductor 2.

(1) two differential current elements; $I_1 \vec{\Delta L_1} = 3\times10^{-6}\hat{a}_y$ Am at $P_1(1,0,0)$ and $I_2\vec{\Delta L_2} = 3\times10^{-6}(-0.5\hat{a}_x+0.4\hat{a}_y+0.3\hat{a}_z)$ Am at $P_2(2,2,2)$ are located in free space. Find the vector force exerted on:

(a) $I_2\vec{\Delta L_2}$ by $I_1\vec{\Delta L_1}$ (b) $I_1\vec{\Delta L_1}$ by $I_2\vec{\Delta L_2}$

Soln:- $d(d\vec{F_2}) = \dfrac{\mu_0 I_1 I_2}{4\pi R_{12}^2} d\vec{L_2}\times(d\vec{L_1}\times\hat{a}_{R_{12}})$

$= \dfrac{\mu_0}{4\pi R_{12}^2} I_2 d\vec{L_2}\times I_1 d\vec{L_1}\times\hat{a}_{R_{12}}$

$= \dfrac{\mu_0}{4\pi R_{12}^2} I_2\vec{\Delta L_2}\times(I_1\vec{\Delta L_1}\times\hat{a}_{R})$

$\vec{R_{12}} = (2-1)\hat{a}_x + 2\hat{a}_y + 2\hat{a}_z$

$\vec{R_{12}} = \hat{a}_x + 2\hat{a}_y + 2\hat{a}_z$

$R_{12} = 3\,m$

$d(d\vec{F_2}) = \dfrac{4\pi\times10^{-7}}{4\pi(3)^2}\Big[3\times10^{-6}(-0.5\hat{a}_x+0.4\hat{a}_y+0.3\hat{a}_z) \times \left(3\times10^{-6}\hat{a}_y\times\left[\dfrac{\hat{a}_x+2\hat{a}_y+2\hat{a}_z}{3}\right]\right)\Big]$

$= \dfrac{10^{-7}}{27}\Big[3\times10^{-6}(-0.5\hat{a}_x+0.4\hat{a}_y+0.3\hat{a}_z) \times (3\times10^{-6}(-\hat{a}_z+2\hat{a}_x))\Big]$

$= \dfrac{10^{-7}}{27}\times9\times10^{-12}\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ -0.5 & 0.4 & 0.3\\ 2 & 0 & -1\end{vmatrix}$

$= \dfrac{10^{-19}}{3}\left[-0.4\hat{a}_x + 0.1\hat{a}_y - 0.8\hat{a}_z\right]$

$= 10^{-20}\left[-\dfrac{4}{3}\hat{a}_x + \dfrac{1}{3}\hat{a}_y - \dfrac{8}{3}\hat{a}_z\right]$

$d(d\vec{F_2}) = (-1.33\hat{a}_x + 0.33\hat{a}_y - 2.67\hat{a}_z)\,10^{-20}\,N$

$d(d\vec{F_1}) = \dfrac{\mu_0 I_1 I_2}{R_{21}^2} d\vec{L_1}\times(d\vec{L_2}\times\hat{a}_{R_{21}})$

$$= \frac{\mu_0 I_1 \vec{\Delta L_1}}{4\pi R_{21}^2} \times (I_2 \vec{\Delta L_2} \times \hat{a}_{R21})$$

$\underline{\underline{R_{21} = 3m}}$

$$= \frac{(4\pi \times 10^{-7})}{4\pi (3)^2} \, 3\times 10^{-6} \hat{a}_y \times \left(3\times 10^{-6}(-0.5\hat{a}_x + 0.4\hat{a}_y + 0.3\hat{a}_z) \times \left(\frac{-\hat{a}_x - 2\hat{a}_y - 2\hat{a}_z}{3}\right)\right)$$

$$= \frac{10^{-7}}{\cancel{27}_3} \times \cancel{9} \times 10^{-12} \hat{a}_y \times \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ -0.5 & 0.4 & 0.3 \\ -1 & -2 & -2 \end{vmatrix}$$

$$= \frac{10^{-19}}{3} \hat{a}_y \times \left[-0.2\hat{a}_x - 1.3\hat{a}_y + 1.4\hat{a}_z\right]$$

$$= \frac{10^{-19}}{3} \left[-0.2(-\hat{a}_z) + 1.4\hat{a}_x\right]$$

$$= \frac{10^{-20}}{3} (2\hat{a}_z + 14\hat{a}_x)$$

$$d(d\vec{F_1}) = \left(\frac{14}{3}\hat{a}_x + \frac{2}{3}\hat{a}_z\right) 10^{-20} \ N$$

$$d(d\vec{F_1}) = \underline{\underline{(4.67\,\hat{a}_x + 0.67\,\hat{a}_z)}}\, 10^{-20} \ N$$

II (Conti from pg 6)

The moment of a force or torque about a specified point is defined as the vector product of the moment arm $\vec{R}$ & the force $\vec{F}$.

$$\therefore \boxed{\vec{T} = \vec{R} \times \vec{F} \ \text{Nm}}$$

Consider a point A at which force $\vec{F}$ is applied as shown in fig (a). Let $\vec{R}$ be the arm from origin O to point A. Then the torque $\vec{T}$ about origin is nothing but a vector product of $\vec{R}$ & $\vec{F}$.

The magnitude of the torque is equal to the product of the magnitudes of $\vec{R}$ & $\vec{F}$ & sine of the angle between $\vec{R}$ & $\vec{F}$, while the direction of the torque $\vec{T}$ is normal to both $\vec{R}$ & $\vec{F}$.

as shown in fig (b) there are two forces $\vec{F_1}$ & $\vec{F_2}$ applied at points A_1 & A_2. The arms for the two forces drawn from the origin be $\vec{R_1}$ & $\vec{R_2}$ respectively.

Assume that $\vec{F_2} = -\vec{F_1}$. Then the total torque $\vec{T}$ about the origin due to the two forces is given by,

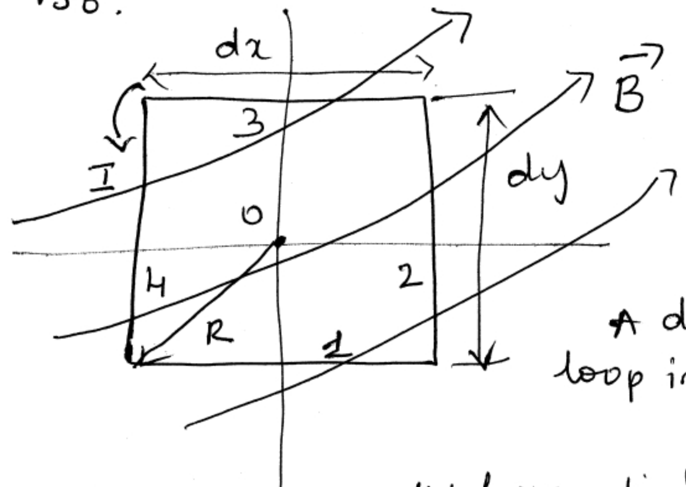
$$\vec{T} = \vec{R_1} \times \vec{F_1} + \vec{R_2} \times \vec{F_2}$$

$$\therefore \vec{T} = (\vec{R_1} - \vec{R_2}) \times \vec{F_1} \qquad [\text{since } \vec{F_2} = -\vec{F_1}]$$

$$\boxed{\vec{T} = \vec{R_{21}} \times \vec{F_1}} \qquad \left[\vec{R_{21}} = \vec{R_1} - \vec{R_2} \text{ is a vector joining } A_2 \text{ to } A_1\right]$$

Magnetic Dipole moment

Consider a differential current loop in a magnetic field $\vec{B}$. The loop lies in the x-y plane. The sides of the loop are parallel to the x & y axis & are of length dx & dy. The value of the magnetic field at the center of the loop is taken as B_0.



A differential current loop in a magnetic field $\vec{B}$

Since the loop is of differential size, the value of $\vec{B}$ at all points on the loop is taken as $\vec{B_0}$. The total force on the loop is therefore zero & we are free to choose the origin for the torque at the center of the loop.

The vector force on side 1 is

$$\vec{dF_1} = I\,dx\,a_x \times \vec{B_0}$$

$$\vec{dF_1} = I\,dx\left(B_{0y}\,\hat{a}_z - B_{0z}\,\hat{a}_y\right)$$

for this side of the loop the lever arm $\vec{R}$ extends from the origin to the midpoint of the side, $\vec{R_1} = -\frac{1}{2}dy\,\hat{a}_y$, & the contribution to the total torque is

$$\vec{dT_1} = \vec{R_1} \times \vec{dF_1}$$

$$= -\frac{1}{2}dy\,\hat{a}_y \times I\,dx\left(B_{0y}\,\hat{a}_z - B_{0z}\,\hat{a}_y\right)$$

$$= -\frac{1}{2}dx\,dy\,I\,B_{0y}\,\hat{a}_x$$

the torque Contribution on Side 3 is found to be same

$$\vec{dT_3} = \vec{R}_3 \times \vec{dF}_3 = \frac{1}{2} dy\, \hat{a}_y \times (-I dx\, \hat{a}_x \times \vec{B}_0)$$

$$= -\frac{1}{2} dx\, dy\, I B_{0y}\, \hat{a}_x = \vec{dT_1}.$$

& $\vec{dT_1} + \vec{dT_3} = -dx\, dy\, I B_{0y}\, \hat{a}_x$

Evaluating the torque on sides 2 & 4

$$\vec{dT_2} + \vec{dT_4} = dx\, dy\, I B_{0x}\, \hat{a}_y$$

& the total torque is then

$$\vec{dT} = I dx\, dy\, (B_{0x}\hat{a}_y - B_{0y}\hat{a}_x)$$

the quantity within the parentheses may be represented by a cross product.

$$\vec{dT} = I dx\, dy\, (\hat{a}_z \times \vec{B}_0)$$

$$\boxed{\vec{dT} = I\, \vec{dS} \times \vec{B}} \qquad \vec{B} = \vec{B}_0.$$

$\vec{dS}$ = vector area of the differential current loop

∴ the product of the loop current & the vector area of the loop as the differential magnetic dipole moment $\vec{dm}$ $A{\cdot}m^2$ thus

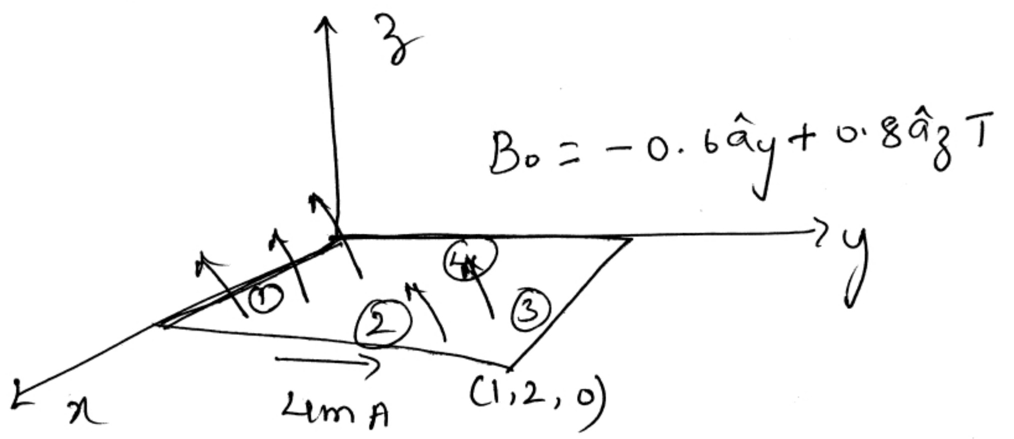
$$\boxed{\vec{dm} = I\, \vec{dS} \quad A{\cdot}m^2}$$

& $\boxed{\vec{dT} = \vec{dm} \times \vec{B}}$

the torque on a planar loop of any size or shape in a uniform magnetic field is given by

$$\boxed{\vec{T} = \vec{I}\vec{S} \times \vec{B} = \vec{m} \times \vec{B}}$$

① Calculate the torque using $\vec{T} = I\vec{S} \times \vec{B}$ for the fig shown below.



Soln:- the loop has dimensions of 1m by 2m & lies in the uniform field

$\vec{B}_0 = -0.6\hat{a}_y + 0.8\hat{a}_z$ T

I (loop current is 4mA).

$\therefore \quad \vec{T} = 4\times10^{-3}\left[(1)(2)\hat{a}_z\right]\times(-0.6\hat{a}_y + 0.8\hat{a}_z)$

$= 4.8\hat{a}_x$ mN.m

thus the loop tends to rotate about an axis parallel to the positive x-axis. The small magnetic field produced by the 4mA loop current tends to line up with $\vec{B}_0$.

② for the fig shown above calculate the total force & torque contribution for each side.

Soln:- on side 1.

$\vec{F}_1 = I\vec{L}_1 \times \vec{B}_0 = 4\times10^{-3}(1\hat{a}_x)\times(-0.6\hat{a}_y + 0.8\hat{a}_z)$

$= -3.2\hat{a}_y - 2.4\hat{a}_z$ mN

On side 3 we obtain the negative of this result

$$\vec{F_3} = 3.2\hat{a}_y + 2.4\hat{a}_z \text{ mN}$$

On side 2.

$$\vec{F_2} = I\vec{L_2} \times \vec{B_o} = 4\times10^{-3}(2\hat{a}_y) \times (-0.6\hat{a}_y + 0.8\hat{a}_z)$$

$$= 6.4\,\hat{a}_x \text{ mN}$$

On side $\vec{F_4} = -6.4\hat{a}_x$ mN

The forces are distributed uniformly along each of the sides, we treat each force as if it were applied at the center of the side. The origin for the torque may be established anywhere since the sum of the forces is zero, i.e. we choose the center of the loop. Thus,

$$\vec{T} = \vec{T}_1 + \vec{T}_2 + \vec{T}_3 + \vec{T}_4 = R_1\times\vec{F_1} + R_2\times\vec{F_2} + R_3\times\vec{F_3} + R_4\times\vec{F_4}$$

$$= (-1\hat{a}_y)\times(-3.2\hat{a}_y - 2.4\hat{a}_z) + (0.5\hat{a}_x)\times(6.4\hat{a}_x)$$

$$+ (1\hat{a}_y)\times(3.2\hat{a}_y + 2.4\hat{a}_z) + (-0.5\hat{a}_x)\times(-6.4\hat{a}_x)$$

$$= 2.4\,\hat{a}_x + 2.4\,\hat{a}_x = \underline{\underline{4.8}}\,\hat{a}_x \text{ mN.m}$$

P.T.O

Magnetic Materials

On the basis of the magnetic behaviour, the magnetic materials are classified.

A charged particle with angular momentum always contributes to the permanent magnetic dipole moments. There are 3 important contributions to the angular moments of the atom.

(a) orbital magnetic dipole moment
(b) Electron spin magnetic moment
(c) Nuclear spin magnetic moment.

In an atom several electrons revolve in the de orbits around the nucleus. This is very much analogous to a small current loop. Quantum no defines the orbital state of motion of an electron in an atom. The first quantum no 'n' indicates principle quantum which determines the energy of an electron. The 2nd quantum no 'l' represents orbital quantum no. The angular momentum of an electron is called spin of an electron. The spin of an electron produces magnetic dipole moment.

In an atom with completely filled orbits the contribution in spin magnetic moment is zero. Similar to the electron spin, the nuclear spin contributes to the magnetic moment called nuclear spin magnetic moment. The mass of the nucleus is much larger than the electron. Thus the dipole moments due to the nuclear spin are very small. The contribution of nuclear magnetic moment to the magnetic properties of the material is negligible.

Classification of magnetic materials. as

(a) Diamagnetic
(b) Paramagnetic
(c) Ferromagnetic
(d) Anti ferromagnetic
(e) Ferrimagnetic
(f) Super magnetic.

(a) Diamagnetic materials :-

The magnetic materials in which the orbital magnetic moment & electron spin magnetic moment cancel each other making net permanent magnetic moment of each atom zero are called diamagnetic materials. Thus with no external field in diamagnetic materials the net torque produced on atom is zero, but an applied field makes spin movement slightly greater than that of orbit movement. This results in small magnetic moment which opposes the applied field. Thus when a diamagnetic material like bismuth is kept near either pole of a strong magnet gets repelled. Other examples of diamagnetic materials are lead, copper, silicon, diamond.

(b) paramagnetic materials

The magnetic materials in which the orbital & spin magnetic moments do not cancel each other resulting in a net magnetic moment of an atom are called paramagnetic materials. In the absence of an external field the paramagnetic materials do not show any magnetic effect. But when an external field is applied each atomic dipole moment experience a torque. Due to this all the atomic dipole moments tend to align with the external field. Hence when the paramagnetic material is kept near the pole of a strong magnet it gets attracted. examples of paramagnetic materials are potassium tungsten, oxygen, rare earth metals.

(c) Ferromagnetic materials

The materials in which the atoms have large dipole moment due to electron spin magnetic moments are called ferromagnetic materials. In such materials the adjacent atoms line up their magnetic dipole moments in parallel fashion in the lattice. The regions in which this alignment happens are called domains. When an external field is applied domains increase their size increasing the internal field to a high value, when the external field is removed some of the moments will remain in a small region which results in residual field which is called hysteresis. ferromagnetic materials eg:- Cobalt, Nickel Iron.

(d) Antiferromagnetic materials

The materials in which the dipole moments of adjacent atoms line up in antiparallel fashion are called antiferromagnetic materials. The net magnetic moment in such materials is zero. Hence when such a material is kept near a strong magnet it gets neither attracted nor repelled. eg:- oxides & chlorides of sulphides at low temperature.

(e) Ferrimagnetic materials

The materials in which the magnetic dipole moments are lined up in antiparallel fashion, but the net magnetic moment is non-zero are called ferrimagnetic materials. The specimen of ferrimagnetic materials gets affected in strong external fields much lower than ferromagnetic materials.

Ex:- Nickel ferrite, Nickel-zinc ferrite, Iron-oxide magnetite.

(f) Super magnetic materials

The important property of supermagnetic materia is that even though each particle of it contains large magnetic domains but they cannot Penetrate adjacent particles.

Ex:- magnetic tapes used for audio, video & data readings.

P.T.O

P.T.O

Magnetization & permeability

The movement of the orbital electrons, electron spin & nuclear spin produce internal magnetic field. The current produced by the bound charges is called bound current I_b. The field produced due to movement of bound charges is called Magnetization ($\vec{M}$)

Let the bound current I_b flows through a closed path. Assume that this closed that encloses a differential area $d\vec{s}$, then the magnetic dipole moment is given by

$$\vec{m} = I_b d\vec{s}$$

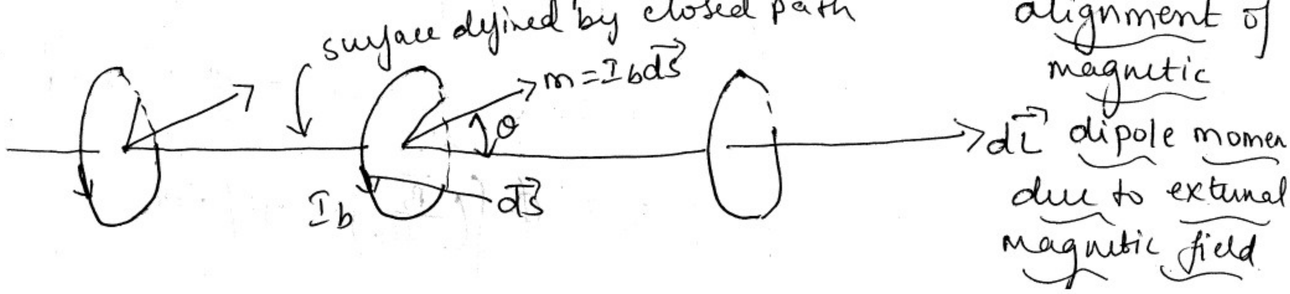
Consider a diff volum Δv & if there are 'n' magnetic dipoles per unit volume, then the total magnetic dipole moment is obtained by summing up all the individual magnetic dipole moment of each magnetic dipole.

$$\therefore \vec{m}_{total} = \sum_{i=1}^{n\Delta v} m_i$$

$\therefore$ Magnetization is defined as the magnetic dipole moment per unit volume A/m.

$$\boxed{\vec{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} m_i \quad A/m.}$$

Now consider a closed path



the magnetic moment $\vec{m}$ makes an angle θ with the element of the closed path $\vec{dL}$. thus its component along the direction of $\vec{dL}$ is nothing but the projection of $\vec{m}$ on $\vec{dL}$. The magnetic dipole moment is perpendicular to the surface area $\vec{ds}$.

∴ Consider a small volume, $\vec{ds} \cos\theta \, dL$ or $\vec{ds} \cdot \vec{dL}$ within which there are $n \vec{ds} \cdot \vec{dL}$ magnetic dipoles, the bound current crossing the surface enclosed by the path has increased by I_b for each of the $n \vec{ds} \cdot \vec{dL}$ dipoles. Thus

$$dI_b = n \, I_b \, \vec{ds} \cdot \vec{dL} = \vec{M} \cdot \vec{dL}$$

& within an entire closed contour,

$$I_b = \oint \vec{M} \cdot \vec{dL} \quad — (a)$$

Eq (a) says that if we go around a closed path & find dipole moments, there will be a corresponding current composed of.

∴ By writing Ampere's circuital law in terms of the total current, bound plus free.

$$\oint \frac{\vec{B}}{\mu_0} \cdot \vec{dL} = I_T$$

Where $\quad I_T = I_b + I$

where I is the total free current enclosed by the closed path.

$$\therefore \quad I = I_T - I_b$$

$$I = \oint \frac{\vec{B}}{\mu_0} \cdot \vec{dL} - \oint \vec{M} \cdot \vec{dL}$$

$$I = \oint \left[\left(\frac{\vec{B}}{\mu_0} \right) - \left(\vec{M} \right) \right] \cdot \vec{dL}$$

We may now define $\vec{H}$ in terms of $\vec{B}$ & $\vec{M}$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

$\vec{B} = \mu_0 \vec{H}$ in free space where magnetization is zero.

$$\therefore \quad \vec{B} = \mu_0 (\vec{H} + \vec{M})$$

or $I = \oint \vec{H} \cdot \vec{dl}$ $\quad \left[\because \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}\right]$

Obtaining ampere's circuital law in terms of the free currents.

$$I_b = \int_S \vec{J_b} \cdot \vec{ds}$$

$$I_T = \int_S \vec{J_T} \cdot \vec{ds}$$

$$I = \int_S \vec{J} \cdot \vec{ds}$$

from Stoke's theorem, we can write

$$\nabla \times \vec{M} = \vec{J_b}$$

$$\nabla \times \frac{\vec{B}}{\mu_0} = \vec{J_T}$$

$$\boxed{\nabla \times \vec{H} = \vec{J}}$$

the Magnetic Susceptibility χ_m can be defined a

$$\boxed{\vec{M} = \chi_m \vec{H}}$$

thus we have, $\vec{B} = \mu_0 (\vec{H} + \chi_m \vec{H})$

$$= \mu_0 \mu_r \vec{H}.$$

where $\mu_r = 1 + \chi_m \rightarrow$ relative permeability.

$\therefore$ permeability $\mu = \mu_0 \mu_r$

$\therefore \boxed{\vec{B} = \mu \vec{H}}$

1) given a ferrite material operating in a linear mode with $B = 0.05T$, $\mu_r = 50$ Calculate values for χ_m, M & H.

Soln:- Since $\mu_r = 1 + \chi_m$

$\chi_m = \mu_r - 1 = \underline{\underline{49}}$

also $\vec{B} = \mu_r \mu_0 \vec{H}$

$$\vec{H} = \frac{0.05}{50 \times 4\pi \times 10^{-7}} = \underline{\underline{796}} \text{ A/m}$$

the magnetization is $M = \chi_m H$ or

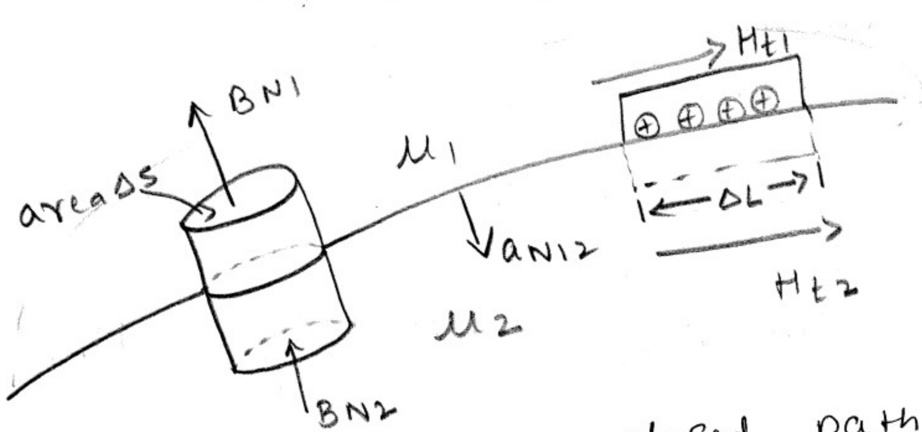
$M = 49 \times 796 = \underline{39000}$ A/m.

Relating $\vec{B}$ & $\vec{H}$.

$\vec{B} = \mu_0 (\vec{H} + \vec{M})$

$0.05 = \underline{\underline{4\pi \times 10^{-7}}} (796 + 39000)$

Magnetic Boundary Conditions



A Gaussian surface & a closed path are constructed at the boundary between media 1 & 2, having permeabilities of μ_1 & μ_2. from this we determine the boundary conditions $B_{N1} = B_{N2}$ & $H_{t1} - H_{t2} = K$.

Fig above shows a boundary between two Isotropic homogeneous linear materials with permeabilities μ_1 & μ_2. The boundary Conditions on the normal Components is determined by allowing the surface to cut a small cylindrical gaussian surface.

Applying Gauss' law for the magnetic field

$$\oint_S \vec{B} \cdot \vec{dS} = 0$$

$$B_{N1} \Delta S - B_{N2} \Delta S = 0$$

or $$\boxed{B_{N2} = B_{N1}}$$

thus $$H_{N2} = \frac{\mu_1}{\mu_2} H_{N1}$$

the normal component of B is continuous, but the normal component of $\vec{H}$ is discontinuous by the ratio μ_1/μ_2.

for linear magnetic materials we can write

$$M_{N2} = \chi_{m2} \frac{\mu_1}{\mu_2} H_{N1} = \frac{\chi_{m2}\mu_1}{\chi_{m1}\mu_2} M_{N1}$$

from ACL

$$\oint \vec{H} \cdot \vec{dL} = I.$$

is applied about a small closed path in a plane normal to the boundary surface. taking a clockwise trip around the path, we get

$$H_{t1}\Delta L - H_{t2}\Delta L = K\Delta L$$

assume that the boundary may carry a surface current K whose component normal to the plane of the closed path is K. thus

$$\boxed{H_{t1} - H_{t2} = K}$$

or

$$(\vec{H}_1 - \vec{H}_2) \times \hat{a}_{N12} = \vec{K}$$

where $\hat{a}_{N12}$ is the unit normal at the boundary directed from region 1 to region 2.

for tangential $\vec{B}$ we have

$$\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K$$

the boundary condition on the tangential component of the magnetization for linear materials is given by

$$M_{t2} = \frac{\chi_{m2}}{\chi_{m1}} M_{t1} - \chi_{m_2} K .$$

P.T.O

Magnetic circuits

In general, in magnetic circuits, we determine the magnetic fluxes and magnetic field intensities in various parts of the circuit. The magnetic circuits are analogous to the electric circuits.

w.k.t the relationship between electric field & electrostatic potential is given by

$$\vec{E} = -\nabla V$$

and we also know that, $\vec{H} = -\nabla V_m$ [Scalar Magnetic potential]

similar to e.m.f (electromotive force) in an electric circuit, we can define a new quantity in a magnetic circuit called magnetomotive force (mmf), unit is Amperes. It is noted that no current may flow in any region in which V_m is defined.

The electric potential difference b/w points A and B can be written as

$$V_{AB} = \int_A^B \vec{E} \cdot \vec{dL} \Rightarrow \text{P.D}$$

& the corresponding relationship b/w mmf & magnetic field intensity is given by

$$V_{mAB} = \int_A^B \vec{H} \cdot \vec{dL}$$

Ohm's law for the electric circuit in point form is,

$$\vec{J} = \sigma \vec{E}$$

|||ly for magnetic circuits $\vec{B} = \mu \vec{H}$ [∵ magnetic flux density is analog of $\vec{J}$]

to find the total current, we must intigrate,

$$I = \int_S \vec{J} \cdot \vec{ds}$$

A corresponding operation is necessary to determine the total magnetic flux flowing through the cross-section of a magnetic circuit.

$$\phi = \int_S \vec{B} \cdot \vec{ds}$$

we then defined resistance as the ratio of potential difference & current or

$$V = IR$$

∴ reluctance is defined as the ratio of the magnetomotive force to the total flux, thus

$$V_m = \Phi \mathcal{R} \quad (A \cdot t/Wb) \quad [\text{Amperes turns}]$$

In resistors which are made of a linear isotropic homogeneous material of conductivity σ & have a uniform cross section of area S & length d, the total resistance is

$$R = \frac{d}{\sigma S}$$

for a linear isotropic homogeneous magnetic material of length d & uniform cross-sections, then the total reluctance is

$$\mathcal{R} = \frac{d}{\mu S}$$

the only such material to which this relationship is air.

Finally, let us consider the analog of the source voltage in an electric circuit. We know that the closed line integral of $\vec{E}$ is zero.

$$\oint \vec{E} \cdot \vec{dL} = 0.$$

in other words, KVL States that the rise in potential through the source is equal to the fall in potential through the load.

The expression for magnetic phenomena takes a slightly different form.

$$\oint \vec{H} \cdot \vec{dL} = I_{total}$$

the total current linked by the path is usually obtained by allowing a current I to flow through on N-turn coil, we may express this as $\oint \vec{H} \cdot \vec{dL} = NI$.

In an electric circuit, the voltage source is a part of the closed path, in the magnetic circuit the current-carrying coil will surround or link the magnetic circuit.

1) An air-core toroid has 500 turns, a cross-sectional area of $6\,cm^2$, a mean radius of 15cm, & a coil current of 4A is available. Find out $\mathcal{R}$, ϕ, $\vec{H}$ & $\vec{B}$.

$I = 4A$
$N = 500$

Soln :- $V_{m,\,source} = NI = 500 \times (4)$

$= 2000\,A\cdot t$

$$\mathcal{R} = \frac{d}{\mu S} = \frac{2\pi(0.15)}{4\pi \times 10^{-7} \times 6 \times 10^{-4}} = 1.25 \times 10^{9}\ A\cdot t/wb$$

$$\Phi = \frac{V_{m,s}}{\mathcal{R}} = \frac{2000}{1.25(10^9)} = 1.6 \times 10^{-6}\ \text{wb}.$$

$$\vec{B} = \frac{\Phi}{S} = \frac{1.6 \times 10^{-6}}{6 \times 10^{-4}} = \underline{\underline{2.67}} \times 10^{-3}\ T$$

$$\vec{H} = \frac{\vec{B}}{\mu} = \frac{2.67 \times 10^{-3}}{4\pi \times 10^{-7}} = 2120\ \text{A.t/m}$$

Potential energy & forces on Magnetic Materials

From the electrostatic theory we can write the general expression for energy is

$$W_E = \frac{1}{2}\int_{vol} \vec{D}.\vec{E}\ dv.$$

A linear relationship b/w $\vec{D}$ & $\vec{E}$ is assumed.

The total energy stored in a steady magnetic field in which $\vec{B}$ is linearly related to $\vec{H}$ is given by

$$W_H = \frac{1}{2}\int_{vol} \vec{B}.\vec{H}\ dv$$

Letting $\vec{B} = \mu\vec{H}$

$$W_H = \frac{1}{2}\int_{vol} \mu H^2\ dv$$

$$W_H = \frac{1}{2}\int_{vol} \frac{B^2}{\mu}\ dv$$

1) A Conductor 6m long, lies along Z-direction with a current of 2A in $\hat{a}_z$ direction. Find the force experienced by conductor if $\vec{B} = 0.08\,\hat{a}_x$ T.

Soln:- A force exerted on current carrying conductor in a magnetic field is given by

$$\vec{F} = I\,\vec{dL} \times \vec{B}$$

$$\vec{F} = 2(6\,\hat{a}_z) \times (0.08\,\hat{a}_x)$$

$$\vec{F} = 12\,\hat{a}_z \times 0.08\,\hat{a}_x$$

$$\vec{F} = 0.96\,\hat{a}_y \text{ N}$$

2) A Conductor of length 2.5m in $z=0$ & $x=4$m carries a current of 12A in $-\hat{a}_y$ direction. Calculate the uniform flux density in the region, if the force on the conductor is 12×10^{-2} N in the direction specified by $\left[\dfrac{-\hat{a}_x + \hat{a}_z}{\sqrt{2}}\right]$

Soln:- Let $\vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$

the force exerted on the conductor is given by

$$\vec{F} = I\,\vec{dL} \times \vec{B}$$

$$\therefore\ 12\times10^{-2}\left[\frac{-\hat{a}_x+\hat{a}_z}{\sqrt{2}}\right] = (-12\,\hat{a}_y)(2.5)\times(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z)$$

$$-(0.0848)\hat{a}_x + (0.0848)\hat{a}_z = -30\left[\hat{a}_y \times (B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z)\right]$$

$$\therefore\ (2.8267\times10^{-3})\hat{a}_x - (2.8267\times10^{-3})\hat{a}_z = -B_x\hat{a}_z + B_z\hat{a}_x$$

Comparing co-efficients ~~of~~ we get

$$B_x = 2.8267 \times 10^{-3}$$

$$B_y = 0 \quad , \quad B_z = 2.8267 \times 10^{-3}$$

thus $\vec{B} = 2.8267 \times 10^{-3} \hat{a_x} + 2.8267 \times 10^{-3} \hat{a_z}$ T

$$\vec{B} = (2.8267 \hat{a_x} + 2.8267 \hat{a_z}) \text{ mT}$$

3) A Conductor 4m long lies along the y-axis with a current of 10.0 A in the $\hat{a_y}$ direction. Find the force on the conductor if the field in the region is in region is $\vec{B} = 0.005 \hat{a_x}$ T.

Soln:- A force exerted a current carrying conductor in the magnetic field $\vec{B}$ is given by

$$\vec{F} = I d\vec{L} \times \vec{B}$$

$$\therefore \vec{F} = [10 (4 \hat{a_y}) \times (0.05 \hat{a_x})]$$

$$= [(10 \times 4 \times 0.05) (\hat{a_y} \times \hat{a_x})]$$

$$= 0.2 (-\hat{a_z}) \text{ N}$$

$\hat{a_y} \times \hat{a_x} = -\hat{a_z}$

Module - 5 - Part-A

Time varying fields & Maxwell's equation

Books Referred:

1) Engineering Electromagnetics - W.H. Hayt, J A Buck 7th edition, Mc Graw-Hill.

2) Field & Wave Electromagnetics - David K Cheng 2nd edition, Pearson Education.

3) Electromagnetics with Applications - Kraus/Fleisch - 5th edition - Mc Graw-Hill

Topics Covered:

Faraday's law
displacement current
Maxwell's equation in point & integral form

Faraday's Law

Introduction

Electric field is produced by a changing magnetic field & a magnetic field is produced by changing Electric field was first demonstrated by Michael Faraday in 1831.

Faraday's Law or Faraday's law of Electromagnetic induction

Statement: "The induced emf around a closed path is equal to the negative of rate of change of the magnetic flux enclosed by that path".

$$\boxed{emf = \frac{-d\Phi}{dt} \text{ volts}}$$

[-ve sign indicates that the direction of the induced emf is such that to produce a current which will produce a magnetic field which will oppose the original field].

the emf is a scalar quantity which is given by

$$emf = \oint_l \vec{E}\cdot\vec{dl} \quad — ②$$

∴ we have

$$\frac{-d\Phi}{dt} = \oint_l \vec{E}\cdot\vec{dl}$$

We have the magnetic flux Φ passing through a specified area is given by

$$\Phi = \int_s \vec{B}\cdot\vec{ds} \quad — ③$$

$$\text{or} \quad \frac{-d}{dt}\int_s \vec{B}\cdot\vec{ds} = \oint_l \vec{E}\cdot\vec{dl}$$

$$\boxed{\text{or} \quad emf = \oint_l \vec{E}\cdot \vec{dl} = -\int_S \frac{\delta \vec{B}}{\delta t}\cdot \vec{ds}}$$ Maxwell's 2nd Eq for time-varying integral-form

stationary induced emf

there are 2 Conditions for the induced emf.

ⓐ If we consider a stationary path, the magnetic flux is the only time varying quantity & taking a partial derive under the integral sign.

from Stoke's theorem

$$\int_S (\nabla \times \vec{E})\cdot \vec{ds} = -\int_S \frac{\delta \vec{B}}{\delta t}\cdot \vec{ds}$$

apply Stoke's theorem to the closed line integral

$$(\nabla \times \vec{E})\cdot \vec{ds} = -\frac{\delta \vec{B}}{\delta t}\cdot \vec{ds}$$ Surface integrals may be taken over identical Surfaces.

$$\boxed{\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t}}$$

Maxwell's 2nd Eq for time varying field in Point form.

ⓑ Moving path within a constant field (motional emf)

Let us Consider a charge Q moving in a magnetic field $\vec{B}$ at a Velocity V, then the force On a charge is given by

$$\vec{F} = Q\,\vec{V}\times\vec{B}$$

the motional electric field intensity is defined as the force per unit charge.

it is given by,

$$\vec{E}_m = \frac{\vec{F}}{Q} = \frac{Q\,\vec{V}\times\vec{B}}{Q} = \vec{V}\times\vec{B}$$

then the induced emf produced by the moving Conductor is given by

$$\boxed{emf = \oint \vec{E}_m\cdot\vec{dl} = \oint_l (\vec{V}\times\vec{B})\cdot\vec{dl}}$$

this emf is called motional or generator emf or flux cutting emf.

If magnetic flux is also changing with time, then the above equation must include both the transfolmer emf and the motional emf.

$$\therefore \boxed{emf = \oint_l \vec{E}\cdot\vec{dl} = -\int_S \frac{\delta\vec{B}}{\delta t}\cdot\vec{ds} + \oint_l (\vec{V}\times\vec{B})\cdot\vec{dl}}$$

total induced emf = transformer emf + motional emf.

(1) given $\vec{B} = B_0 e^{kt}\hat{a}_z$. Find $\vec{E}$ using both point and integral form of maxwell's equation within cylindrical region $\rho < b$ in $z=0$ plane with Only E_ϕ Component by symmetry.

Soln:- Consider a circular path $\rho = a$ & $a<b$ along with E_ϕ must be Constant by Symmetry

from, Faraday's Law

$$emf = \oint_l \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{ds} \quad \Big\}\text{ maxwells Eq in integral form}$$

$$\int_0^{2\pi} E_\phi \hat{a}_\phi \cdot \rho d\phi \, \hat{a}_\phi = -\int \frac{\delta \vec{B}}{\delta t} \cdot \vec{ds}$$

$$E_\phi \rho \phi \Big]_0^{2\pi} = -\int_s \frac{\delta \vec{B}}{\delta t} \cdot \vec{ds}$$

$\rho = a$

$$E_\phi 2\pi a = -\int_s B_o K e^{Kt} \hat{a}_z \cdot \rho d\phi d\rho \hat{a}_z$$

$$= -\int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} B_o K e^{Kt} \rho d\rho d\phi$$

$$= -B_o K e^{Kt} \frac{\rho^2}{2} \Big]_0^a 2\pi$$

$$E_\phi 2\pi a = -\frac{2\pi B_o K e^{Kt} a^2}{2}$$

$$E_\phi = \frac{-\cancel{2\pi} B_o K e^{Kt} a^{\cancel{2}}}{\cancel{2\pi} \times \cancel{a} \times 2}$$

replace 'a' with ρ

$$E_\phi = -\frac{1}{2} B_o K e^{Kt} \rho \hat{a}_\phi \ \ V/m$$

Maxwell's eqn.

$$\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t}$$

$$\nabla \times \vec{E} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \delta/\delta\rho & \delta/\delta\phi & \delta/\delta z \\ 0 & \rho E_\phi & 0 \end{vmatrix}$$

$$= \frac{1}{\rho} \frac{\delta}{\delta \rho} (\rho E_\phi) \hat{a}_z + \frac{1}{\rho} \left[-\frac{\delta}{\delta z} (\rho E_\phi) \hat{a}_\rho \right.$$

$\llcorner$ (1)

& $$-\frac{\delta \vec{B}}{\delta t} = -K B_0 e^{Kt} \hat{a}_z \quad \text{—(2)}$$

Eq (1) & (2) comparing

$$\frac{1}{\rho}\frac{\delta}{\delta \rho}(\rho E_\phi) = -K B_0 e^{Kt}$$

$$\frac{\delta}{\delta \rho}(\rho E_\phi) = -\rho K B_0 e^{Kt}$$

$$\rho E_\phi = -\frac{\rho^2}{2} B_0 K e^{Kt}$$

$$E_\phi = \frac{-K}{2} \rho^2 B_0 e^{Kt}$$

$$\vec{E} = -\frac{K}{2} \rho B_0 e^{Kt} \hat{a}_\phi \ \text{V/m}$$

---

2) within a certain region $\varepsilon = 10^{-11}$ F/m & $\mu = 10^{-5}$ H/m if $B_x = 2 \times 10^{-4} \cos 10^5 t \sin 10^{-3} y$ Tesla.

(a) use $\nabla \times \vec{H} = \varepsilon \frac{d\vec{E}}{dt}$ to find $\vec{E}$.

$\vec{B} = \mu \vec{H}$

$$\therefore \frac{1}{\mu}(\nabla \times \vec{B}) = \frac{1}{\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ B_x & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{\mu} \hat{a}_z \left(-\frac{\delta}{\delta y} B_x\right)$$

$$= -\frac{1}{\mu} 2 \times 10^{-7} \times \cos 10^5 t \cos 10^{-3} y \, \hat{a}_z$$

$$\varepsilon \frac{\delta \vec{E}}{\delta t} = \frac{-2 \times 10^{-7}}{10^5} \cos 10^5 t \cos 10^{-3} y \, \hat{a}_z$$

$$\vec{E} = \frac{-2\times10^{-2}\sin 10^5 t \cos 10^{-3}y \; \hat{a}_z}{10^5 \times 10^{-11}}$$

$$\vec{E} = -20,000 \sin 10^5 t \cos 10^{-3} y \; \hat{a}_z \text{ A/m}$$

(b) find the total magnetic flux passing the surface $x=0$, $0<y<40\text{m}$, $0<z<2\text{m}$ at $t = 1\mu s$.

$$\Phi = \int_s \vec{B}_x \cdot \vec{ds} = \int_{y=0}^{40}\int_{z=0}^{2} 2\times10^{-4}\cos 10^5 t \sin 10^{-3}y \, dy\,dz$$

$$= 2\times10^{-4}\cos\left[10^5(1\times10^{-6})\right]\left[-\frac{\cos 10^{-3}y}{10^{-3}}\right]_0^{40}\left[z\right]_0^2$$

$$= 0.318 \text{ mwb}$$

(c) find the value of closed line integral of $\vec{E}$ around the perimeter.

$$\oint_e \vec{E}\cdot\vec{dl} = \int_s \frac{-\delta\vec{B}}{\delta t}\cdot\vec{ds}$$

$$= \int_{y=0}^{40}\int_{z=0}^{2} \frac{-\delta\vec{B}}{\delta t}\,dy\,dz$$

$$= -2\times10^{-4}\left[-\sin 10^5(1\times10^{-6})\right]\times10^5\left[\frac{-\cos 10^{-3}y}{10^{-3}}\right]_0^{40}\left[z\right]_0^2$$

$$= 3.19\text{V}$$

$\vec{J}_D = 0.0150 \text{ A/m}^2$

(b) In the air space at a point within large power distribution transformer.

$\vec{B} = 0.8 \cos[1.257 \times 10^{-6}(3 \times 10^8 t - x)]\hat{a}_y$ T.

$$\vec{I}_D = \frac{\delta \vec{D}}{\delta t} = \nabla \times \vec{H} = \frac{1}{\mu}\nabla \times \vec{B}$$

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ 0 & B_y & 0 \end{vmatrix} = \hat{a}_z\left(\frac{\delta}{\delta x}B_y\right)$$

$$= \hat{a}_z \frac{\times 0.8}{\mu_0}\left[-\sin(1.257 \times 10^{-6}(3 \times 10^8 t - x) \times -1.257 \times 10^{-6}\right]$$

$$= \frac{0.8 \times 1.257 \times 10^{-6}}{4\pi \times 10^{-7}} = 0.80023 \text{ A/m}^2$$

(d) In a metallic conductor at 60 Hz, if $\varepsilon = \varepsilon_0$ & $\mu = \mu_0$

$\sigma = 5.8 \times 10^7$ S/m, & $\vec{J} = \sin(377t - 117.1z)\hat{a}_x$ MA/m

Soln: $\vec{J} = \sigma \vec{E}$

$\vec{E} = \vec{J}/\sigma$

$$\vec{E} = \frac{1}{5.8 \times 10^7}\sin(377t - 117.1z)\hat{a}_x \text{ MV/m}$$

$$\therefore \vec{J}_D = \frac{\delta \vec{D}}{\delta t} = \varepsilon_0 \frac{\delta \vec{E}}{\delta t}$$

$$= \frac{8.854 \times 10^{-12}}{5.8 \times 10^7} \times 377\left[\cos(377t - 117.1z)\hat{a}_x \times 10^6\right]$$

$$\vec{J}_D = 57.55 \text{ pA/m}^2$$

3) find the amplitude of the displacement current density.

(a) Adjacent to an automobile antenna where the magnetic field intensity of an FM Signal is

$$H_x = 0.15 \cos\left[3.12\left(3\times10^8 t - y\right)\right] \ A/m$$

Soln:- for a non-conducting medium $\rho_v = 0, \sigma = 0$

$$\therefore \quad \nabla \times \vec{H} = \frac{\delta \vec{D}}{\delta t}$$

$$\vec{I_d} = \frac{\delta \vec{D}}{\delta t} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ H_x & H_y & H_z \end{vmatrix}$$

$$= \hat{a}_x\left[\frac{\delta}{\delta y}H_z - \frac{\delta}{\delta z}H_y\right] - \hat{a}_y\left[\frac{\delta}{\delta x}H_z - \frac{\delta}{\delta z}H_x\right] + \hat{a}_z\left[\frac{\delta}{\delta x}H_y - \frac{\delta}{\delta y}H_x\right]$$

$$= -\hat{a}_z \frac{\delta}{\delta y} H_x .$$

$$= -\hat{a}_z \times 0.15 \times -\sin\left[3.12\left(3\times10^8 t - y\right)\right] \times (3.12)$$

$$|\vec{I_d}| = 0.468 \ A/m^2$$

(b) within a large air filled power capacitor where $\varepsilon_r = 5\varepsilon_0$ $\vec{E} = 0.9 \cos\left[1.257\times10^{-6}\left(3\times10^8 t - 2\sqrt{5}\right)\right] \hat{a}_x \ MV/m$

soln:

$$\vec{J_D} = \frac{\delta \vec{D}}{\delta t} = \varepsilon \frac{\delta \vec{E}}{\delta t} = \varepsilon_0 \varepsilon_r 0.9\left[-\sin\left(1.257\times10^{-6}\left(3\times10^8 t - 2\sqrt{5}\right)\right)\times 1.257\times10^{-6}\times 3\times10^8\right] \hat{a}_x$$

* Displacement current

From Ampere's circuital law for steady magnetic fields,

$$\nabla \times \vec{H} = \vec{J} \quad \text{—— (1)}$$

taking divergence on both the sides,

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} \quad \text{—— (2)}$$

but "divergence of the curl of any vector field is zero".

$$\therefore \quad \nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} = 0 \quad \text{—— (3)}$$

But the equation of Continuity is given by

$$\nabla \cdot \vec{J} = -\frac{\delta \rho_v}{\delta t} \quad \text{—— (4)}$$

from equation (4) it is clear that when

$$-\frac{\delta \rho_v}{\delta t} = 0 .$$

then equation (3) becomes true.
but not compatible or true for time - varying fields.

$\therefore$ We must modify eq (1) by adding one unknown term say $\vec{G}$.

$$\therefore \quad \nabla \times \vec{H} = \vec{J} + \vec{G} \quad \text{—— (5)}$$

again taking divergence on both the sides,

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} + \nabla \cdot \vec{G} = 0$$

$$\nabla \cdot \vec{G} = -\nabla \cdot \vec{J}$$

$$= \frac{\delta \rho_v}{\delta t} = \frac{\delta}{\delta t}(\nabla \cdot \vec{D}) \quad \text{from Gauss's Law } \boxed{\nabla \cdot \vec{D} = \rho_v}$$

$$\nabla \cdot \vec{G} = \nabla \cdot \frac{\delta \vec{D}}{\delta t}$$

Comparing two Sides of the eqns.

$$\boxed{\vec{G} = \frac{\delta \vec{D}}{\delta t}}$$

Hence Ampere's circuital law in point form for time varying fields can be written as

$$\nabla \times \vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t} \quad \text{—— (6)}$$

$\frac{\delta \vec{D}}{\delta t} \Rightarrow$ time varying electric flux density & is called displacement current density denoted by $\vec{J_D}$.

$$\therefore \boxed{\nabla \times \vec{H} = \vec{J} + \vec{J_D}} \quad \text{— (7)}$$

∴ the total displacement current crossing any given surface is

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t}}$$

integrating over the surface

$$\int_s \nabla \times \vec{H} \, \vec{ds} = \int_s \vec{J} \cdot \vec{ds} + \int_s \frac{\delta \vec{D}}{\delta t} \cdot \vec{ds} \quad \text{—— (8)}$$

we have Stokes theorem

$$\oint_l \vec{H} \cdot \vec{dl} = \int_s (\nabla \times \vec{H}) \cdot \vec{ds} \quad \text{—— (9)}$$

Comparing Eq (8) & (9).

$$\oint_l \vec{H}\cdot\vec{dl} = \int_s \vec{J}\cdot\vec{ds} + \int_s \frac{\delta\vec{B}}{\delta t}\cdot\vec{ds}$$

$$\boxed{\oint_l \vec{H}\cdot\vec{dl} = I + \int_s \frac{\delta\vec{D}}{\delta t}\cdot\vec{ds}} \quad — (10)$$

$$\oint_l \vec{H}\cdot\vec{dl} = I + I_d \quad — (11)$$

Maxwell's eqn in integral form.

I_d = total displacement current. - which is the current produced by time varying Electric field,

$$I = \int_s \vec{J}\cdot\vec{ds} \quad \& \quad I_d = \int_s \frac{\delta\vec{D}}{\delta t}\cdot\vec{ds}.$$

Conduction current density is $\vec{J} = \sigma\vec{E}$ is the motion of charges (usually $\bar{e}$) in a region of zero net charge density.

Conventional current density is $\vec{J} = \rho_v\vec{V}$ is the motion of volume charge density.

(1) Let $\mu = 10^{-5}$ H/m, $\varepsilon = 4 \times 10^{-9}$ F/m, $\sigma = 0$, $\rho_v = 0$. Find K & So that each of the following pairs of fields Satisfies Maxwell's eqns.

(a) $\vec{D} = 6a\hat{x} - 2y\hat{a_y} + 2z\hat{a_z}$ nc/m²

$\vec{H} = Kx\hat{a_x} + 10y\hat{a_y} - 25z\hat{a_z}$ A/m

Soln:- from maxwell's eqn

$\nabla \cdot \vec{D} = \rho_v$

$\nabla \cdot \vec{D} = \frac{\delta}{\delta x} D_x + \frac{\delta}{\delta y} D_y + \frac{\delta}{\delta z} D_z$

$= 0 - 2 + 2 = 0$ since $\rho_v = 0$ it is true.

& from one more maxwell's eqn.

$\nabla \cdot \vec{B} = 0$

$\nabla \cdot \mu\vec{H} = 0 \qquad \nabla \cdot \vec{H} = 0$

$\frac{\delta}{\delta x}\vec{H_x} + \frac{\delta}{\delta y} H_y + \frac{\delta}{\delta z} \vec{H_z} = 0$

$K + 10 - 25 = 0$

$K = \underline{\underline{15 \text{ A/m}^2}}$

Maxwells Equations in Point & Integral form.

(1) Maxwell's first equation

it is obtained from the Faraday's law according to "Faraday's law of electromagnetic induction"

$$\oint_l \vec{E} \cdot \vec{dl} = -\int_S \frac{\delta}{\delta t} \vec{B} \cdot \vec{ds} \quad \text{—— (1)}$$ Maxwells Eq in integral form.

Using Stokes theorem, We Can write

$$\oint_l \vec{E} \cdot \vec{dl} = \int_S \nabla \times \vec{E} \cdot \vec{ds} \quad \text{— (2)}$$

Comparing Eq (1) & Eq (2)

$$\int_S (\nabla \times \vec{E}) \cdot \vec{ds} = -\int_S \frac{\delta \vec{B}}{\delta t} \cdot \vec{ds}$$

$$\boxed{\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t}}$$ Maxwells Eq in point form.

(2) Maxwell's 2nd Eq :-

it is obtained from Ampere's circuital law.

$$\oint_l \vec{H} \cdot \vec{dl} = I_{enclosed} \quad \text{— (1)}$$

where $I_{enc} = I + I_d \rightarrow$ total current.

where $I = \int_S \vec{J} \cdot \vec{ds} \rightarrow$ Conduction current

$I_d = \int_S \vec{J_D} \cdot \vec{ds} \rightarrow$ displacement current.

$$\boxed{\oint_l \vec{H} \cdot \vec{dl} = I + \int_S \frac{\delta \vec{D}}{\delta t} \cdot \vec{ds}} \quad \text{— (2)}$$ Maxwells Eq in integral form.

Using Stokes theorem

$$\oint_l \vec{H} \cdot \vec{dl} = \int_S \nabla \times \vec{H} \cdot \vec{ds} \quad \text{— (3)}$$

Comparing Eq (2) & (3)

$$\int_S (\nabla \times \vec{H}) \cdot \vec{ds} = \int_S \left(J + \frac{\delta \vec{D}}{\delta t}\right) \cdot \vec{ds}$$

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t}}$$ Maxwells Eq in Point form.

(3) Maxwells third equation

obtained from Gauss's law applied to "electricfield".

from Gauss's law $\boxed{\oint_S \vec{D}\cdot\vec{ds} = Q_{enclosed}}$ — (1)

where $Q_{enclo} = \int_{vol} \rho_v \, dv$

$$\therefore \boxed{\oint_S \vec{D}\cdot\vec{ds} = \int_{vol} \rho_v \, dv} \quad — (2)$$

using divergence theorem,

$$\oint_S \vec{D}\cdot\vec{ds} = \int_{vol} (\nabla\cdot\vec{D}) \, dv \quad — (3)$$

comparing Eq (2) & (3)

$$\int_{vol} (\nabla\cdot\vec{D}) dv = \int_{vol} \rho_v \, dv$$

$\boxed{\nabla\cdot\vec{D} = \rho_v}$ Maxwell's Eq in point form.

(4) maxwell's 4th equation

obtained from Gauss's law applied to "magnetic field"

W.K.T $\oint_S \vec{B}\cdot\vec{ds} = 0$ —— (1)

using divergence theorem.

$$\oint_S \vec{B}\cdot\vec{ds} = \int_{vol} (\nabla\cdot\vec{B}) \, dv \quad —— (2)$$

Comparing Eq (1) & (2)

$$\int_{vol} (\nabla\cdot\vec{B}) \, dv = 0$$

$\boxed{\nabla\cdot\vec{B} = 0}$ Maxwell's Eq in point form.

1) Show that the ratio of the magnitudes of Conduction current density to displacement current density is equal to $\frac{\sigma}{j\omega\varepsilon}$

Soln:- we have Conduction current density (from Point form of Ohm's law).

$$\vec{J} = \sigma \vec{E} \quad —(1)$$

$\vec{E}$ = where $\vec{E}$ is the Electric field intensity

& the displacement current density is

$$\vec{J}_D = \frac{\delta \vec{D}}{\delta t} \quad —(2)$$

$\vec{D}$ = Electric flux density = $\varepsilon \vec{E}$

$$\vec{J}_D = \frac{\delta(\varepsilon \vec{E})}{\delta t} \quad —(3)$$

for a time varying field

$$\vec{E} = \vec{E}_0 e^{j\omega t} \quad —(4)$$

using Eq (4) in Eq (1) & in Eq (3)

$$\vec{J} = \sigma \vec{E}_0 e^{j\omega t} \quad —(5)$$

$$\vec{J}_D = \frac{\delta}{\delta t}(\varepsilon \vec{E}_0 e^{j\omega t})$$

$$= \varepsilon \vec{E}_0 j\omega e^{j\omega t} = j\omega \varepsilon \vec{E}_0 e^{j\omega t} \quad —(6)$$

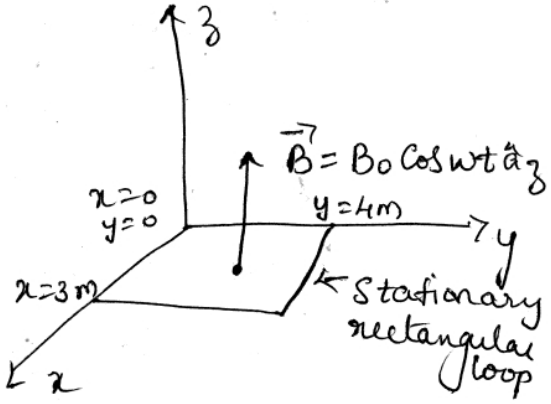
divide Eq (5) by Eq (6)

$$\frac{\vec{J}_c}{\vec{J}_D} = \frac{\sigma \vec{E}_0 e^{j\omega t}}{j\omega \varepsilon \vec{E}_0 e^{j\omega t}}$$

$$\boxed{\frac{\vec{J}_c}{\vec{J}_D} = \frac{\sigma}{j\omega\epsilon}}$$ Proved

2) A stationary rectangular loop is placed in the Plane $z=0$ as shown in fig. If the magnetic flux density is $\vec{B} = B_0 \cos\omega t \, \hat{a}_z$, then calculate the induced emf around the loop.

Soln:-
Here loop is stationary & the magnetic flux density varies with time, so emf is given by



$$emf = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{ds}$$

$$= -\frac{d}{dt}\int_S (B_0 \cos\omega t \, \hat{a}_z \cdot dx\,dy\,\hat{a}_z) \quad , \quad (\because B = B_0 \cos\omega t \, \hat{a}_z)$$

$$= -\frac{d}{dt}\int_{y=0}^{4}\int_{x=0}^{3} B_0 \cos\omega t \, dx\,dy$$

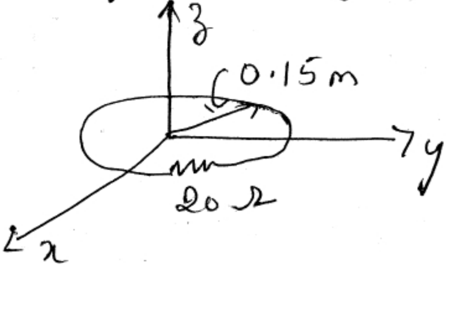
$$= -\frac{d}{dt} B_0 \cos\omega t \, [x]_0^3 \, [y]_0^4$$

$$= -\frac{d}{dt}(12 B_0 \cos\omega t)$$

$$emf = 12\,\omega B_0 \sin\omega t \text{ Volts}$$

3) The circular loop conductor having a radius of 0.15 m is placed in the XY-plane. This loop consists of a resistance of 20 Ω as shown. If the magnetic flux density is $\vec{B} = 0.5 \sin 10^3 t \, \hat{a}_z$. Then find out the current flowing through this loop.

Soln:- Here loop is stationary & the magnetic flux density varies with time,
So emf is given by



$$emf = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{ds}$$

$$= -\frac{d}{dt}\int_S (0.5\sin 10^3 t\,\hat{a}_z \cdot \pi(0.15)^2\hat{a}_z)$$

$$= \frac{-d}{dt}\int_S (0.5\sin 10^3 t\,\hat{a}_z \cdot 0.0225\pi\,\hat{a}_z)$$

$$= -\frac{d}{dt}(0.01125\pi\sin 10^3 t)$$

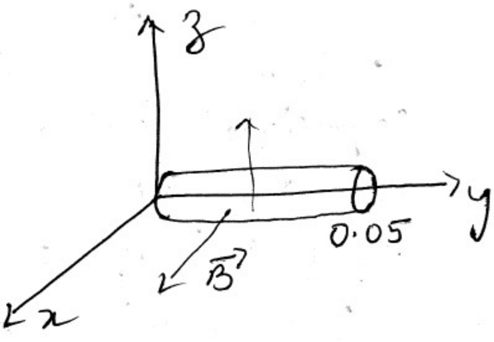
$$emf = -\underline{\underline{11.25\pi}}\cos 10^3 t \text{ Volts}$$

∴ current flowing through the loop is,

$$I = \frac{emf}{R} = \frac{-11.25\pi\cos 10^3 t}{20}$$

$$I = -0.\underline{\underline{5625}}\pi\cos 10^3 t$$

4) A Conductor is placed along y-axis as shown. This Conductor moves with the velocity of $\vec{V} = 5\cos 10^3 t\,\hat{a}_z$. If $\vec{B} = 0.06\,\hat{a}_x$ Tesla, find the induced voltage in the Conductor.



Soln:- Here Conductor is moving & the magnetic field is time invariant, So emf is given by,

$$emf = \oint_l (\vec{V}\times\vec{B})\cdot\vec{dL}$$

$$= \int_{0.5} [(5\cos 10^3 t\,\hat{a}_z)\times(0.06\,\hat{a}_x)]\cdot dy\,\hat{a}_y$$

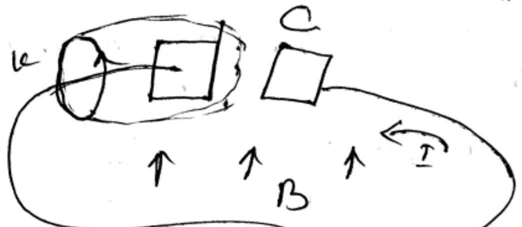
$$= \int_{y=0}^{0.5} (-0.3\cos 10^3 t\,\hat{a}_y)\cdot(dy\,\hat{a}_y)$$

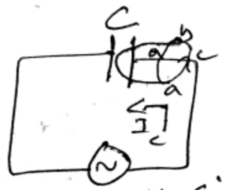
$$= -0.3\cos 10^3 t\int_0^{0.5} dy$$

$$= -0.3\cos 10^3 t\,[y]_0^{0.5}$$

$$emf = -\underline{\underline{0.15\cos 10^3 t}} \text{ Volts}$$

Show that the Conduction current is equal to the displacement current.





$V = V_0 \sin \omega t$.

Say a Capacitor is Connected to an ac source & the Capacitor plates are Separated by a dielectric material. It can be Shown that Continuity of Current on two Sides of the Capacitor is maintained by the displacement current between the plates.

$\therefore$ ~~let~~ the Voltage across the Capacitor is

$$V = V_0 \sin \omega t.$$

$$\therefore \quad Q = CV = CV_0 \sin \omega t$$

$$I_c = \frac{dQ}{dt} = \frac{d}{dt} CV_0 \sin \omega t$$

$$\boxed{I_c = C\omega V_0 \cos \omega t} \quad -(1)$$

assume the Capacitor is a lossless parallel plate Capacitor where the Spacing between the plates is d, area of each plate is A & permittivity of medium $= \varepsilon$

& assuming A to be large compared to d. the Electric field intensity b/w the plates is

$$E = \frac{V}{d} = \frac{V_0 \sin \omega t}{d} \Rightarrow D = \varepsilon E = \varepsilon \frac{V}{d} = \frac{\varepsilon V_0 \sin \omega t}{d}$$

$$\vec{J}_D = \frac{\delta D}{\delta t} = \frac{\delta}{\delta t}\left[\frac{\varepsilon}{d} V_0 \sin \omega t\right]$$

$$I_d = A \cdot \frac{\delta D}{\delta t} = A \frac{\delta}{\delta t}\left[\frac{\varepsilon}{d} V_0 \sin \omega t\right] = \frac{A \varepsilon \omega V_0 \cos \omega t}{d}$$

$$\boxed{I_d = C\omega V_0 \cos \omega t} \quad -(2) \qquad \boxed{I_c = I_D}$$

thus the conduction current is equal to the displacement current (i.e, current flowing within the capacitor).

Note :-

(1) Maxwell's equations for time - varying fields

Point or differential form	Integral form	Derived from
$\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t}$	$\oint \vec{E} \cdot \vec{dL} = -\int_S \frac{\delta \vec{B}}{\delta t} \cdot \vec{ds}$	Faraday's law
$\nabla \times \vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t}$	$\oint \vec{H} \cdot \vec{dL} = I + \int_S \frac{\delta \vec{D}}{\delta t} \cdot \vec{ds}$	Ampere's circuital law
$\nabla \cdot \vec{D} = \rho_v$	$\oint_S \vec{D} \cdot \vec{ds} = \int_{vol} \rho_v \, dv$	Gauss's law (electric field)
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot \vec{ds} = 0$	Gauss's Law (Magnetic field)

(2) Maxwell's equations for static fields (time-invariant)

Point or Differential form	Integral form
$\nabla \times \vec{E} = 0$	$\oint \vec{E} \cdot \vec{dL} = 0$
$\nabla \times \vec{H} = \vec{J}$	$\oint \vec{H} \cdot \vec{dL} = \int_S \vec{J} \cdot \vec{ds}$
$\nabla \cdot \vec{D} = \rho_v$	$\oint_S \vec{D} \cdot \vec{ds} = \int_{vol} \rho_v dv$
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot \vec{ds} = 0$

Module 5 - part - B
Uniform plane Wave

Topics Covered

General wave Equation
wave propagation in free Space
Wave propagation in good Conductor
Skin Effect
Poynting's theorem & wave power.

Books Referred

1. "Engineering Electromagnetics", William H Hayt Jr, & John A Buck, Tata - Mc - Graw Hill 7th edition, 2006.
2. "Electromagnetic waves & Radiating systems", Edward C. Jordan & Keith G Balmain, Prentice - Hall of India/ pearson Education. 2nd Edition.
3. "principles of Electromagnetics" — By SC Mahapatra & Sudipta Mahapatra Mc-Graw Hill.

Introduction

Maxwell's equation for time varying fields form the basics for understanding the Electromagnetic wave propagation.

Waves are the means of transporting energy or information from source to destination. The waves consisting of electric & magnetic fields are called electromagnetic waves.

Wave is a function of time and space

Eg:- Radio waves, light rays etc.

Wave: Physical Phenomenon that occurs at one Place at a given time is repeated at other Place at later time. The time delay depends upon the space separation between the two Points Considered.

A ~~plane~~ wave is to be plane wave, if the Phase of the wave is same at all points on the Plane Surface. If the amplitude is also same at all points on the plane surface, it is said to be an uniform plane wave.

To consider wave motion in free space Maxwell's equations may be written in terms of $\vec{E}$ & $\vec{H}$ only as

$$\nabla \times \vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t}$$

$$\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t}$$

$$\nabla \cdot \vec{D} = \rho_v$$

$$\nabla \cdot \vec{B} = 0$$

A uniform plane wave is one in which both fields $\vec{E}$ and $\vec{H}$ lie in the transverse plane i.e, (the plane whose normal is the direction of propagation and both fields are of constant magnitude in transverse plane.

Hence they are called TEM waves
transverse Electromagnetic waves.

Eg:- $\vec{E} = E_x \hat{a}_x$

if Electric field is polarized in x-direction but wave travel in z-direction, there is spatial variation of $\vec{E}$ only with z.

with these restrictions, curl of $\vec{E}$ reduces to

$$\nabla \times \vec{E} = \frac{\delta \vec{E}_x}{\delta z} \hat{a}_y = -\mu_0 \frac{\delta \vec{H}}{\delta t} = -\mu_0 \frac{\delta H_y}{\delta t} \hat{a}_y$$

the direction of curl of $\vec{E}$ determines the direction of $\vec{H}$, hence $\vec{H}$ is along y-direction. & $\vec{H}$ varies only in z-direction.

$$\therefore \nabla \times \vec{H} = -\frac{\delta H_y}{\delta z} \hat{a}_x =$$

$\therefore$ $\vec{E}$ & $\vec{H}$ direction of wave travel mutually orthogonal.

General Wave Equations

Wave equations can be obtained by relating the space and time variations of the electric and magnetic fields, using Maxwell's equation.

from Maxwell's equation

$$\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t} = -\mu \frac{\delta \vec{H}}{\delta t}$$

taking curl on the above equation

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\delta}{\delta t}(\nabla \times \vec{H}) \quad \text{—— (1)}$$

W.K.T from maxwell's equation

$$\nabla \times \vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t} = \sigma \vec{E} + \varepsilon \frac{\delta \vec{E}}{\delta t}$$

substituting above Equation in Eq (1)

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\delta}{\delta t}\left(\sigma \vec{E} + \varepsilon \frac{\delta \vec{E}}{\delta t}\right)$$

$$\left[\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}\right]$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu\sigma \frac{\delta \vec{E}}{\delta t} - \mu\varepsilon \frac{\delta^2 \vec{E}}{\delta t} \quad \text{—— (2)}$$

Considering sourceless, $\rho_v = 0$

from maxwell's Equation.

$$\nabla \cdot \vec{D} = \rho_v$$

$$\nabla \cdot \vec{D} = \varepsilon \, \nabla \cdot \vec{E} = 0$$

Hence equation (2) becomes,

$$\boxed{\nabla^2 \vec{E} = \mu\sigma \frac{\delta \vec{E}}{\delta t} + \mu\varepsilon \frac{\delta^2 \vec{E}}{\delta t^2}}$$

multiplying both sides by ε

$$\boxed{\nabla^2 \vec{D} = \mu\sigma \frac{\delta \vec{D}}{\delta t} + \mu\varepsilon \frac{\delta^2 \vec{D}}{\delta t}}$$

Wave Equation for $\vec{E}$ & $\vec{D}$

from Maxwell's equation,

$$\nabla\times\vec{H} = \vec{J} + \frac{\delta\vec{B}}{\delta t}$$

taking curl on both sides.

$$\nabla\times\nabla\times\vec{H} = \sigma\,\nabla\times\vec{E} + \varepsilon\,\nabla\times\frac{\delta\vec{E}}{\delta t}$$

$$\nabla(\nabla\cdot\vec{H}) - \nabla^2\vec{H} = -\sigma\mu\frac{\delta\vec{H}}{\delta t} - \varepsilon\mu\frac{\delta^2\vec{H}}{\delta t^2}$$

[since $\nabla\cdot\vec{B} = \mu\nabla\cdot\vec{H} = 0$]

$\vec{J} = \sigma\vec{E}$

$\vec{D} = \varepsilon\vec{E}$

$\nabla\times\vec{E} = -\frac{\delta\vec{B}}{\delta t}$

$\nabla\times\vec{E} = -\mu\frac{\delta\vec{H}}{\delta t}$

$\varepsilon\frac{\delta\nabla\times\vec{E}}{\delta t} = \varepsilon\frac{\delta}{\delta t}\mu\frac{\delta\vec{H}}{\delta t}$.

$$\boxed{\nabla^2\vec{H} = \mu\sigma\frac{\delta\vec{H}}{\delta t} + \mu\varepsilon\frac{\delta^2\vec{H}}{\delta t^2}} \quad \text{— Eq (a)}$$

$$\therefore \boxed{\nabla^2\vec{B} = \mu\sigma\frac{\delta\vec{B}}{\delta t} + \mu\varepsilon\frac{\delta^2\vec{B}}{\delta t^2}} \quad \text{— Eq (b)}$$

Wave equation for $\vec{H}$ & $\vec{B}$

[above equation (a) ×by by μ] we get equation (b).

Wave propagation in free Space (loss-less Medium)

A dielectric medium is one in which the conduction current is almost zero such a medium may be treated as a perfect dielectric or lossless medium ($\sigma = 0$).

Let $\vec{E}$ is in y-direction

$\vec{H}$ is in z- direction

and hence wave propagates in x- direction & all the three are mutually ⊥r to each other. Thus $\vec{E}$ & $\vec{H}$ are functions of 'x' and 't'.

$$\therefore \vec{E} = E_y(x,t)\,\hat{a}_y$$

$$\vec{H} = H_z(x,t)\,\hat{a}_z$$

By letting $\sigma = 0$, the wave equations for $\vec{E}$ and $\vec{H}$ becomes.

$$\nabla^2 \vec{E} = \mu \varepsilon \frac{\delta^2 \vec{E}}{\delta t^2}$$

$$\nabla^2 \vec{H} = \mu \varepsilon \frac{\delta^2 \vec{H}}{\delta t^2}$$

Expanding $\nabla^2 \vec{E}$ in rectangular Co-ordinate system the above equation becomes.

$$\mu \varepsilon \frac{\delta^2 \vec{E}}{\delta t^2} = \frac{\delta^2 \vec{E}}{\delta x^2} + \frac{\delta^2 \vec{E}}{\delta y^2} + \frac{\delta^2 \vec{E}}{\delta z^2}$$

Since wave propagates in 'x' direction and it is a function of 'x' and 't' only the last two terms are zero.

(Let $\vec{E} = E_y \hat{a}_y$)

$$\therefore \mu \varepsilon \frac{\delta^2 E_y}{\delta t^2} = \frac{\delta^2 \vec{E_y}}{\delta x^2}$$

$$\frac{\delta^2 \vec{E_y}}{\delta x^2} = \mu \varepsilon \frac{\delta^2 E_y}{\delta t^2} \quad \text{— (1)}$$

Let $E_y = E_0 e^{j\omega t}$

then $\frac{\delta^2 E_y}{\delta t^2} = E_0 (j\omega)^2 e^{j\omega t}$ [differentiating partially w.r.t 't']

$$\frac{\delta^2 E_y}{\delta x^2} = -w^2 E_0 e^{j\omega t} \mu \varepsilon$$

$\therefore$ Eq (1) can be written as.

$$\frac{\delta^2 E_y}{\delta x^2} + w^2 E_0 e^{j\omega t} = 0$$

$$\frac{\delta^2 E_y}{\delta x^2} + \mu \varepsilon \omega^2 E_y = 0 \quad \text{—} (2)$$

Let $\frac{\delta^2}{\delta x^2} = m^2$

$$m^2 E_y + \mu \varepsilon \omega^2 E_y = 0$$

$$(m^2 + \mu \varepsilon \omega^2) E_y = 0$$

Hence the characteristics equation is

$$m^2 + \mu \varepsilon \omega^2 = 0$$

$$m = \pm j \omega \sqrt{\mu \varepsilon}$$

$$m = \pm j \beta \qquad \Big| \beta = \omega \sqrt{\mu \varepsilon}$$

Where $\beta = \omega \sqrt{\mu \varepsilon}$ is phase shift Constant in radians/m.

Hence the solution of second-order linear Constant Coefficient differential equation is

$$E_y(x) = C_1 e^{-j\beta x} + C_2 e^{j\beta x} \quad \text{(in general)}$$

$$E_y(x) = E_{ym} e^{-j\beta x} + E'_{ym} e^{j\beta x} \quad \text{—} (3)$$

E_{ym} & E'_{ym} are constants w.r.t 'x'.

the above equation is the phasor of frequency domain form of electric field which has a sinusoidal variation at a fixed frequency.

To find the time domain field, we multiply the phasor result of equation (3) by $e^{j\omega t}$ and take the real part of it.

$$E_y(x,t) = Re\left\{ E_{ym} e^{-j\beta x} e^{j\omega t} + E'_{ym} e^{j\beta x} e^{j\omega t} \right\}$$

$$= Re\left\{ E_{ym} e^{j(\omega t - \beta x)} + E'_{ym} e^{j(\omega t + \beta x)} \right\}$$

$$E_y(x,t) = E_{ym} \underbrace{\cos(\omega t - \beta x)}_{\text{forward travelling wave (+x-direction) (incident wave)}} + E'_{ym} \underbrace{\cos(\omega t + \beta x)}_{\text{Backward travelling wave (-x-direction) (Reflected wave)}} \quad -④$$

Phase Velocity (V_p)

The velocity V_p of plane wave is the velocity with which the phase of the wave propagates. It ensures as time varies the phase of the cosine function remains unchanged.

To find the phase velocity of a forward travelling wave.

for a forward-travelling wave,

$$\omega t - \beta x = m \qquad [m = \text{Const}]$$

differentiate wrt 't' we get.

$$\omega - \beta \frac{dx}{dt} = 0$$

Where $V_p = \frac{dx}{dt}$

$$V_p = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

thus the phase velocity of a forward travelling wave $\boxed{V_p = \frac{1}{\sqrt{\mu\varepsilon}}}$

$$V_p = \frac{1}{\sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$\mu_r = \varepsilon_r = 1$ for free space

Where $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8$ m/s.

So in free space wave travels with the velocity of light as μ_r & ε_r are both unity.

Wavelength

$$\lambda = \frac{V_p}{f} = \frac{\omega}{\beta f}$$

$\omega = 2\pi f$

$$\boxed{\lambda = \frac{2\pi}{\beta}}$$

wavelength - the distance that must be travelled by the wave to change phase by 2π radians

Relationship between $\vec{E}$ & $\vec{H}$ in a perfect dielectric medium

Letting the wave propagating in x-direction

$$\vec{E} = E_y \hat{a}_y$$

$$\vec{H} = H_z \hat{a}_z$$

from Maxwell's equation

$$\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t} = -\mu \frac{\delta \vec{H}}{\delta t}$$

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ E_x & E_y & E_z \end{vmatrix} = -\mu \frac{\delta}{\delta t}(H_z \hat{a}_z)$$

Since the wave propagates in x-direction the above equation simplifies to,

$$\begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \delta/\delta x & 0 & 0 \\ 0 & E_y & 0 \end{vmatrix} = -\mu \frac{\delta}{\delta t} H_z \hat{a_z}$$

$\frac{\delta}{\delta y}$ & $\frac{\delta}{\delta z}$ are zero, since $\vec{E}$ is a function of only 'x' and 't'. E_x & H_x, E_z & H_y are zero becoz of the transverse nature of the uniform plane wave.

$$\therefore \frac{\delta E_y}{\delta x} \hat{a_z} = -\mu \frac{\delta H_z}{\delta t} \hat{a_z}$$

$$\frac{\delta E_y}{\delta x} = -\mu \frac{\delta H_z}{\delta t}$$

W.K.T $\quad E_y(x,t) = E_{ym} \cos(\omega t - \beta x) + E'_{ym} \cos(\omega t + \beta x)$

then $\frac{\delta E_y}{\delta x} = \beta E_{ym} \sin(\omega t - \beta x) - \beta E'_{ym} \sin(\omega t + \beta x)$

$$\beta E_{ym} \sin(\omega t - \beta x) - \beta E'_{ym} \sin(\omega t + \beta x) = -\mu \frac{\delta H_z}{\delta t}$$

i.e, $\frac{\delta H_z}{\delta t} = -\frac{\beta}{\mu} E_{ym} \sin(\omega t - \beta x) + \frac{\beta}{\mu} E'_{ym} \sin(\omega t + \beta x)$

integrating the above equation w.r.t 't' we get,

$$H_z = \frac{\beta}{\mu \omega} E_{ym} \cos(\omega t - \beta x) - \frac{\beta}{\mu \omega} E'_{ym} \cos(\omega t + \beta x) \quad \text{(i)}$$

$|||^{ly}$ the wave equation for $\vec{H}$ field is

$$\nabla^2 \vec{H} = \mu \varepsilon \frac{\delta^2 \vec{H}}{\delta t^2}$$

$$H_z(x,t) = H_{zm} \cos(\omega t - \beta x) + H'_{zm} \cos(\omega t + \beta x) \quad \text{--- (2)}$$

Comparing Eq (1) & (2)

$$H_{zm} = \frac{\beta}{\mu \omega} E_{ym} = \frac{E_{ym}}{\left(\frac{\mu\omega}{\beta}\right)} = \frac{E_{ym}}{\sqrt{\frac{\mu}{\varepsilon}}} \quad \left[\because \beta = \omega\sqrt{\mu\varepsilon}\right]$$

or $$H'_{zm} = -\frac{\beta}{\mu\omega} E'_{ym} = \frac{-E'_{ym}}{\left(\frac{\omega\mu}{\beta}\right)} = \frac{-E'_{ym}}{\sqrt{\frac{\mu}{\varepsilon}}}$$

$$\boxed{\therefore \left|\frac{\vec{E}}{\vec{H}}\right| = \frac{\vec{E}}{\vec{H}} = \frac{E_y}{H_z} = \eta = \sqrt{\frac{\mu}{\varepsilon}} \ \Omega}$$

$\therefore$ impedance of free space.

$$\boxed{\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377 \Omega}$$

for a medium other than free space, the intrinsic impedance is,

$$\boxed{\eta = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}} = 377 \sqrt{\frac{\mu_r}{\varepsilon_r}} \ \Omega}$$

Wave propagation in a Lossy medium (Conducting)

Let us now consider a uniform plane wave travelling in a medium where the conductivity is non-zero, $\sigma \neq 0$ i.e, a lossy medium.

$$\nabla^2 \vec{E} = \mu\sigma \frac{\delta \vec{E}}{\delta t} + \mu\varepsilon \frac{\delta^2 \vec{E}}{\delta t^2}$$

$$\nabla^2 \vec{H} = \mu\sigma \frac{\delta \vec{H}}{\delta t} + \mu\varepsilon \frac{\delta^2 \vec{H}}{\delta t^2}$$

$$\frac{\delta^2 \vec{E}}{\delta x^2} + \frac{\delta^2 \vec{E}}{\delta y^2} + \frac{\delta^2 \vec{E}}{\delta z^2} = \mu\sigma \frac{\delta \vec{E}}{\delta t} + \mu\varepsilon \frac{\delta^2 \vec{E}}{\delta t^2}.$$

By considering the wave propagation in x-direction

$$\vec{E} = E_y \hat{a}_y$$

$$\frac{\delta^2 \vec{E}_y}{\delta x^2} = \mu\varepsilon \frac{\delta^2 E_y}{\delta t^2} + \mu\sigma \frac{\delta E_y}{\delta t} \quad \text{— (1)}$$

Let $E_y = E_0 e^{j\omega t}$

differentiating w.r.t 't'

$$\frac{\delta E_y}{\delta t} = j\omega E_0 e^{j\omega t} = j\omega E_y$$

$$\frac{\delta^2 E_y}{\delta t^2} = (j\omega)^2 E_0 e^{j\omega t} = (j\omega)^2 E_y \quad \text{— (2)}$$

$$\frac{\delta^2 E_y}{\delta x^2} = \mu\sigma j\omega E_y + \mu\varepsilon (j\omega)^2 E_y$$

$$= j\omega\mu(\sigma + j\omega\varepsilon)E_y$$
$$= \gamma^2 E_y$$

$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

γ = propagation Const.

$$\gamma = \alpha + j\beta$$

γ - is a Complex no having a real part α

α - attenuation Const. & $\beta \rightarrow$ phase shift Constant rad/m.

the characteristic equation for the linear Constant Co-efficient partial differential equation is,

$$m^2 - \gamma^2 = 0$$
$$m = \pm\gamma$$

Hence the solution of partial differential equation is,

$$\frac{\delta^2 E_y}{\delta x^2} = \gamma^2 E_y \text{ is}$$

$$E_y = Re\left[E_{ym} e^{-\gamma x} + E'_{ym} e^{\gamma x}\right]$$

$$E_y(x,t) = Re\left[E_{ym} e^{-\alpha x} e^{-j\beta x} e^{j\omega t} + E'_{ym} e^{\alpha x} e^{j\beta x} e^{j\omega t}\right]$$

$$E_y(x,t) = E_{ym} e^{-\alpha x} \cos(\omega t - \beta x) + E'_{ym} e^{\alpha x} \cos(\omega t + \beta x)$$

III^{ly} Wave equation with $\vec{H}$ is

$$H_z(x,t) = Re\left\{\left[\frac{E_{ym}}{\eta} e^{-\alpha x} e^{-j\beta x} + \frac{E'_{ym}}{\eta} e^{\alpha x} e^{j\beta x}\right\} e^{j\omega t}\right]$$

$$= \frac{E_{ym}}{|\eta|} e^{-\alpha x} \cos(\omega t - \beta x - \theta_n) + \frac{E'_{ym}}{|\eta|} e^{\alpha x} \cos(\omega t + \beta x - \theta_n)$$

the intrinsic impedance of a lossy medium is a complex number and is given by.

$$\eta = |\eta| \angle \theta_n$$

the structure of the solution for lossy media are identical to those of lossless media and consists of both forward - travelling and back-ward travelling waves.

The term $E_{ym} e^{-\alpha x}$ & $E'_{ym} e^{\alpha x}$ amplitudes of the forward & backward travelling waves.

The term $e^{\pm\alpha x}$ cause the amplitude of the forward and backward travelling waves to decay as wave propagates through the medium.

Hence α is called the attenuation constant.

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta.$$

γ = propagation const → is a measure of the change undergone by the amplitude of wave as it propagates in a medium its a complex quantity.

To find expression for the intrinsic impedance of a lossy media

From Maxwell's equation

$$\nabla \times \vec{E} = -\frac{\delta \vec{B}}{\delta t}$$

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ E_x & E_y & E_z \end{vmatrix} = -\mu \frac{\delta}{\delta t}[H_x \hat{a}_x + H_y \hat{a}_y + H_z \hat{a}_z]$$

as the wave propagates in 'x' direction the above eq simplifies to

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \delta/\delta x & 0 & 0 \\ 0 & E_y & 0 \end{vmatrix} = -\mu \frac{\delta H_z}{\delta t} \hat{a}_z$$

$\frac{\delta}{\delta y}$ & $\frac{\delta}{\delta z}$ are zero, since $\vec{E}$ is a function of only 'x' & 't'. E_x, E_z, H_x & H_y are zero because of the transverse nature of the uniform plane wave.

$$\therefore \quad \frac{\delta E_y}{\delta x} \hat{a}_z = -\mu \frac{\delta H_z}{\delta t} \hat{a}_z$$

$$\frac{\delta E_y}{\delta x} = -\mu \frac{\delta H_z}{\delta t} \quad \text{—} (1)$$

the electric field intensity at any point on x-axis at any time in a general medium for the incident wave is given by

$$E_y = E_y(x,t) = E_{ym}\, e^{-\gamma x} e^{j\omega t} \quad \text{—} (2)$$

Differentiating Eq (2) partially w.r.t x

$$\frac{\delta E_y}{\delta x} = E_{ym}\, e^{j\omega t} \cdot e^{-\gamma x}(-\gamma) \quad \text{--- (3)}$$

Substituting Eq (3) in Eq (1)

$$E_{ym}(-\gamma)\, e^{-\gamma x}\, e^{j\omega t} = -\mu \frac{\delta H_z}{\delta t}$$

$$\frac{\delta H_z}{\delta t} = \frac{\gamma}{\mu} E_{ym}\, e^{j\omega t} \cdot e^{-\gamma x}$$

Integrating w.r.t 't'

$$H_z = \frac{\gamma}{\mu} E_{ym} \frac{e^{j\omega t}}{j\omega} e^{-\gamma x}$$

$$H_z = \frac{\gamma}{j\omega\mu} E_{ym}\, e^{j\omega t} e^{-\gamma x}$$

$$H_z = \frac{\gamma}{j\omega\mu} E_y$$

$$\frac{E_y}{H_z} = \frac{j\omega\mu}{\gamma}$$

$$\frac{|E_y|}{|H_z|} = \frac{E}{H} = \frac{E_y}{H_z} = \frac{j\omega\mu}{\gamma}$$

$$\frac{E}{H} = \eta = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}}$$

$$= \frac{\cancel{\sqrt{j\omega\mu}} \cdot \sqrt{j\omega\mu}}{\cancel{\sqrt{j\omega\mu}} \cdot \sqrt{\sigma + j\omega\varepsilon}} = \boxed{\sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}\ \Omega = \eta}$$

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon\left[1 + \frac{\sigma}{j\omega\varepsilon}\right]} \times \frac{j}{j}} = \sqrt{\frac{\mu}{\varepsilon} \cdot \frac{1}{\left(1 - \frac{j\sigma}{\omega\varepsilon}\right)}}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left[1 - \frac{j\sigma}{\omega\varepsilon}\right]^{-1/2}$$

expanding using binomial expansion &
Considering 1^{st} two terms of series

$$\boxed{\eta = \sqrt{\frac{\mu}{\varepsilon}}\left[1 + \frac{j\sigma}{2\omega\varepsilon}\right]}$$

binomial expansion

$$(1+x)^n = 1 + \frac{nx}{1!} + n(n-1)\frac{x^2}{2!}$$

$$x = -\frac{j\sigma}{\omega\varepsilon}$$

$$n = -\frac{1}{2}$$

$$n = 1 + \left(-\frac{1}{2}\right)\left(-\frac{j\sigma}{\omega\varepsilon}\right)$$

$$\eta = |n| \angle \theta_n$$

Relationship between ω & β

$$V_p = \frac{\omega}{\beta}$$

$$\lambda = \frac{2\pi}{\beta}.$$

Loss tangent

$$\frac{\sigma}{\omega\varepsilon} \quad \text{or} \quad \frac{\sigma}{\omega} = \varepsilon'', \quad \varepsilon = \varepsilon'$$

$\frac{\varepsilon''}{\varepsilon'}$ is called loss tangent

loss tangent is the criterion to judge the loss, whether the loss is small or not.

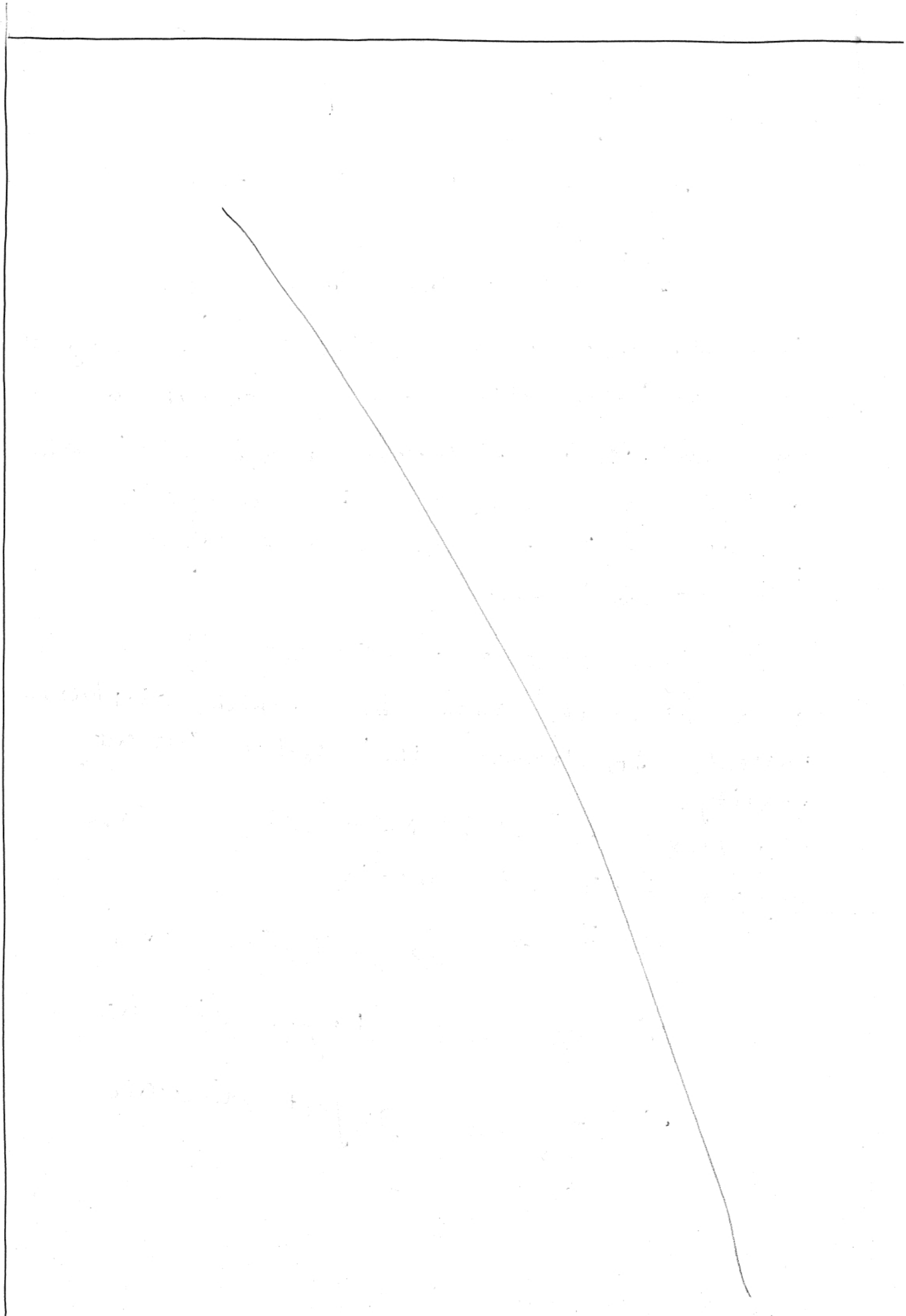
eg:- Consider a dielectric material, in which loss is very small, the magnitude of loss tangent has direct influence on attenuation const α.

$\therefore$ loss tangent $= \tan\theta = \frac{\sigma}{\omega\varepsilon} = \frac{\varepsilon''}{\varepsilon'}$

where θ is the angle by which displacement current J_D leads the total current density.

$\therefore$ based on this ratio $\frac{\sigma}{\omega\varepsilon}$ we can differentiate the media

1. $\frac{\sigma}{\omega\varepsilon} \gg$ good conductor
2. $\frac{\sigma}{\omega\varepsilon} << 1$, Perfect dielectric
3. $\frac{\sigma}{\omega\varepsilon} = 0$, perfect dielectric



Propagation in a good Conductor (Skin effect)

loss tangent $\frac{\sigma}{\omega\varepsilon} \gg 1$ or $\frac{\varepsilon''}{\varepsilon'} \gg 1$; high loss.

A good Conductor has a higher Conductivity & large conduction current. The energy represented by the wave travelling through a material decreases as the wave propagates.

W.K.T the general expression for propagation Constant is,

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\alpha + j\beta = \sqrt{j\omega\mu\sigma - \omega^2\mu\varepsilon}$$

$$= \sqrt{j\omega\mu\sigma\left(1 - \frac{\omega^2\mu\varepsilon}{j\omega\mu\sigma}\right)}$$

$$= \sqrt{j\omega\mu\sigma\left(1 + \frac{j\omega\varepsilon}{\sigma}\right)}$$

$\frac{\sigma}{\omega\varepsilon} \gg 1$

since $\frac{\omega\varepsilon}{\sigma} \ll 1$

$$= \sqrt{j\omega\mu\sigma}$$

$+j = 90^\circ$

$\sqrt{j} = \sqrt{90^\circ}$

$j = 45$

$$\alpha + j\beta = \sqrt{\mu\omega\sigma\,\angle 90^\circ}$$

$$= \sqrt{\omega\mu\sigma}\ \angle 45^\circ$$

$$= \sqrt{\omega\mu\sigma}\,(\cos 45^\circ + j\sin 45^\circ)$$

$$= \sqrt{\frac{\omega\mu\sigma}{2}} + j\sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}} \quad \& \quad \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

in a good conducting medium, attenuation const & phase shift const are equal.

Skin depth δ

is defined as the depth at which the amplitude of wave is attenuated to $\frac{1}{e}$ or 0.368 or 36.7% of the initial amplitude.

If only field E_y component travelling in $+x$-direction,

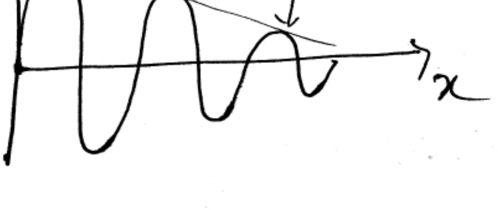
$$E_y = E_{ym} e^{-\alpha x} \cos(\omega t - \beta x)$$

the exponential factor is unity at $x=0$, & reduces to $1/e$ when $x = 1/\alpha$ Skin depth $=\delta$

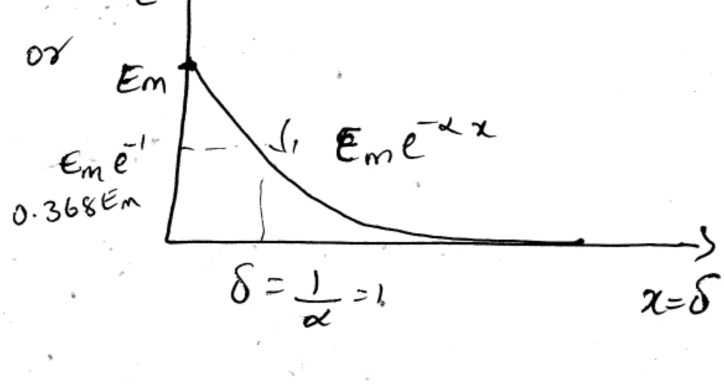
$$\therefore \delta = \frac{1}{\alpha} = \frac{1}{\beta} = \frac{1}{\sqrt{\frac{\mu\omega\sigma}{2}}} = \sqrt{\frac{2}{\mu\omega\sigma}}$$

$$\delta = \sqrt{\frac{2}{2\pi f \mu\sigma}} = \sqrt{\frac{1}{\pi f \mu\sigma}} = \delta$$

it is an important parameter in describing conductor behaviour in electromagnetic fields.



or



All fields in a good conductor such as copper are essentially zero at distance greater than a few skin depths from the surface. Any current density / electric field established at the surface of a good conductor decays rapidly as we propagates into the conductor.

Electromagnetic energy is not transmitted in the interior of a conductor, it travels in the region surrounding the conductor, while the conductor merely guides the wave.

Since $\beta = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{\beta} = \boxed{2\pi\delta = \beta}$

$$V_p = \frac{\omega}{\beta} \quad \text{or} \quad \boxed{V_p = \omega\delta}$$

intrinsic impedance.

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{\sigma\left(1 + \frac{j\omega\varepsilon}{\sigma}\right)}}$$

$$= \sqrt{\frac{j\omega\mu}{\sigma}}$$

$\left[\because 1 + \frac{j\omega\varepsilon}{\sigma} \simeq 1\right.$

$\frac{\sigma}{\omega\varepsilon} \gg 1$

$$= \sqrt{\frac{\omega\mu}{\sigma}\angle 90^\circ}$$

$$\boxed{\eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^\circ \; \Omega}$$

Poynting's theorem

By means of Electromagnetic waves energy can be transported from transmitter to receiver as we know $\vec{E}$ & $\vec{H}$ are basic fields. $\vec{E}$ is expressed in V/m & $\vec{H}$ is expressed in A/m. The product of 2 fields are dimentionally equal to W/m^2. This product of $\vec{E}$ & $\vec{H}$ gives a new quantity called power density.

In order to find power flow associated with an electromagnetic wave it is necessary to develop a power theorem for EM field known as Poynting theorem.

from Maxwells curl equation

$$\nabla \times \vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t} \quad — ①$$

$$\vec{J} = \nabla \times \vec{H} - \frac{\delta \vec{D}}{\delta t}$$

take scalar product on both sides with $\vec{E}$.

$$\vec{E} \cdot \vec{J} = \vec{E} \cdot (\nabla \times \vec{H}) - \vec{E} \cdot \frac{\delta \vec{D}}{\delta t} \quad — ②$$

from a vector identity

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}) \quad — ③$$

$$\vec{E} \cdot (\nabla \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{H}) \quad — ④$$

Hence, substituting Eq ④ in Eq ② we get.

$$\vec{E}\cdot\vec{J} = \vec{H}\cdot(\nabla\times\vec{E}) - \nabla\cdot(\vec{E}\times\vec{H}) - \vec{E}\cdot\frac{\delta\vec{D}}{\delta t}$$

$$\nabla\times\vec{E} = -\frac{\delta\vec{B}}{\delta t}$$

$$= -\vec{H}\cdot\frac{\delta\vec{B}}{\delta t} - \vec{E}\cdot\frac{\delta\vec{D}}{\delta t} - \nabla\cdot(\vec{E}\times\vec{H})$$

$$= -\mu\vec{H}\cdot\frac{\delta\vec{H}}{\delta t} - \varepsilon\vec{E}\cdot\frac{\delta\vec{E}}{\delta t} - \nabla\cdot(\vec{E}\times\vec{H}) \quad \text{—} (5)$$

the two derivative can be arranged as,

$$\varepsilon\vec{E}\cdot\frac{\delta\vec{E}}{\delta t} = \frac{1}{2}\frac{\delta}{\delta t}(\vec{D}\cdot\vec{E}) \quad \text{—} (6)$$

$$\because \varepsilon\left[\vec{E}\cdot\frac{\delta\vec{E}}{\delta t}\right] = \varepsilon\left[E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z\right]\cdot\frac{\delta}{\delta t}\left[E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z\right]$$

$$= \varepsilon\left[E_x\frac{\delta}{\delta t}E_x + E_y\frac{\delta}{\delta t}E_y + E_z\frac{\delta}{\delta t}E_z\right]$$

Let $E_x = \cos\omega t \Rightarrow E_x^2 = \cos^2\omega t$

$$\frac{\delta}{\delta t}E_x^2 = 2\cos\omega t\cdot(-\sin\omega t)\cdot\omega$$

$$\frac{\delta}{\delta t}E_x^2 = 2E_x\frac{\delta}{\delta t}E_x.$$

$$= \frac{\varepsilon}{2}\frac{\delta}{\delta t}\left[E_x^2 + E_y^2 + E_z^2\right]$$

$$= \frac{1}{2}\frac{\delta E^2}{\delta t} = \frac{1}{2}\frac{\delta}{\delta t}\vec{D}\cdot\vec{E}.$$

$||^{ly}$ $$\mu\vec{H}\cdot\frac{\delta\vec{H}}{\delta t} = \frac{\mu}{2}\frac{\delta H^2}{\delta t} = \frac{1}{2}\frac{\delta}{\delta t}(\vec{B}\cdot\vec{H}) \quad \text{—} (7)$$

Hence substitute Eq (6) & (7) in Eq (5)

$$\vec{E}\cdot\vec{J} = -\frac{1}{2}\frac{\delta}{\delta t}(\vec{B}\cdot\vec{H}) - \frac{1}{2}\frac{\delta}{\delta t}(\vec{D}\cdot\vec{E}) - \nabla\cdot(\vec{E}\times\vec{H})$$

$$-\nabla\cdot(\vec{E}\times\vec{H}) = \vec{E}\cdot\vec{J} + \frac{1}{2}\frac{\delta}{\delta t}(\vec{D}\cdot\vec{E}) + \frac{1}{2}\frac{\delta}{\delta t}(\vec{B}\cdot\vec{H})$$

taking volume integral on both sides.

$$-\int_{vol}\nabla\cdot(\vec{E}\times\vec{H})dv = \int_{vol}\vec{E}\cdot\vec{J}\,dv + \frac{1}{2}\frac{\delta}{\delta t}\int_{vol}\vec{D}\cdot\vec{E}\,dv + \frac{1}{2}\frac{\delta}{\delta t}\int_{vol}\vec{B}\cdot\vec{H}\,dv$$

Changing volume integral to surface integral in LHS.

$$\boxed{-\oint_s(\vec{E}\times\vec{H})\cdot\vec{ds} = \int_{vol}\vec{E}\cdot\vec{J}\,dv + \frac{\delta}{\delta t}\int_{vol}\frac{1}{2}\vec{D}\cdot\vec{E}\,dv + \frac{\delta}{\delta t}\int_{vol}\frac{1}{2}\vec{B}\cdot\vec{H}\,dv} \quad (8)$$

the above equation (8) is known as Poynting's theorem.

the terms on the RHS is the total power flowing into this Volume. & LHS total power flowing out of the volume. is,

$$\oint_s(\vec{E}\times\vec{H})\cdot\vec{ds} \text{ watts}$$

the cross product is known as Poynting vector 's':

$$\boxed{\vec{S} = \vec{E}\times\vec{H}} \quad W/m^2 \text{ powerdensity}$$

the direction of $\vec{S}$ indicates the direction of instantaneous flow of power at a point. Hence $\vec{S}$ is normal to both $\vec{E}$ & $\vec{H}$.

Poynting's theorem states that

$\vec{E} \times \vec{H}$ at any point is the measure of energy flow or power flow per unit area at that point.

Let the Electromagnetic wave be propagated along x- direction.

Let $\vec{E} = E_y \hat{a}_y$

$\vec{H} = H_z \hat{a}_z$

In a perfect dielectric (lossless), the $\vec{E}$ & $\vec{H}$ amplitudes are

$$E_y = E_{ym} \cos(\omega t - \beta x)$$

$$H_z = \frac{E_{ym}}{\eta} \cos(\omega t - \beta x)$$

$\therefore$ power density

$$\vec{S_x} = \vec{E} \times \vec{H} = \vec{E}_{ym} \times \left(\frac{E_{ym}}{\eta}\right) \cos(\omega t - \beta x)$$

$$S_x = \frac{E_{ym}^2}{\eta} \cos^2(\omega t - \beta x) \quad W/m^2$$

In case of lossy dielectric,

$$E_y = E_{ym} e^{-\alpha x} \cos(\omega t - \beta x)$$

$$H_z = \frac{E_{ym}}{|\eta|} e^{-\alpha x} \cos(\omega t - \beta x - \theta_n) \qquad \eta = |\eta| \angle \theta_n$$

thus,

$$S_x = E_y H_z = \frac{E_{ym}^2}{\eta} e^{-2\alpha x} \cos(\omega t - \beta x - \theta_n) \cos(\omega t - \beta x)$$

$$S_x = \frac{E_{ym}^2}{\eta} \frac{1}{2} \left[\cos(2\omega t - 2\beta x - \theta_n) + \cos\theta_n\right] e^{-2\alpha x}$$

Problems

① Express $E_y(z,t) = 100\cos(10^8 t - 0.5z + 30^\circ)$ as a phasor.

Soln:-
$$E_y(z,t) = 100\cos(10^8 t - 0.5z + 30^\circ)$$
$$= Re\left[100\, e^{j(10^8 t - 0.5z + 30^\circ)}\right]$$

drop Re part & suppress $e^{j10^8 t}$.

Phasor $E_y(z) = 100 e^{-j0.5z + j30^\circ}$

② The electric field amplitude of a uniform plane wave propagating in the $\hat{a}_z$ direction is 250 V/m. If $\vec{E} = E_x \hat{a}_x$ & $\omega = 1M$ rad/sec find ⓐ frequency ⓑ wavelength ⓒ period ⓓ Amplitude of $\vec{H}$.

Soln:-

(a) $W = 2\pi f$

$$f = \frac{W}{2\pi} = \frac{1\times10^6}{2\pi} = 159\,KHz$$

(b) $$\lambda = \frac{c}{f} = \frac{3\times10^8}{159\times10^3} = 1.88\,Km$$

(c) period, $$T = \frac{1}{f} = \frac{1}{159K} = 6.28\,\mu s$$

(d) amplitude of $\vec{H}$.

$$\eta = \frac{|\vec{E}|}{|\vec{H}|}$$

$$|\vec{H}| = \frac{|\vec{E}|}{\eta} = \frac{250}{377} = 0.663\,A/m$$

---

(3) A plane EM wave is propagating in Z-direction in a dielectric medium of relative permittivity $\varepsilon_r = 5$. The electric field is in $-x$-direction & has an rms value of 0.1 V/m. What is the direction & magnitude of the magnetic field? Calculate the frequency of the wave. Given $\lambda = 5\,m$.

Soln:-

Wave travelling in z direction

$\therefore$ EF is in x-direction

MF is in y-direction

0.1 V/m - rms value

So peak value $= 0.1\times\sqrt{2}$

given $E_x = (0.1\times\sqrt{2})\cos(wt - \beta z)$ V/m

$$\eta = \frac{E_x}{H_y}$$

where $$\eta = 377\sqrt{\frac{\mu_r}{\varepsilon_r}} = 377\sqrt{\frac{1}{5}} = 168.59\,\Omega$$

$$H_y = \frac{0.1 \times \sqrt{2}}{168.59} \cos(\omega t - \beta z)$$

$$H_y = \underline{8.388} \times 10^{-4} \cos(\omega t - \beta z) \text{ A/m}.$$

velocity of propagation,

$$V_p = \frac{1}{\sqrt{\mu_r \varepsilon_r}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{\mu_r \varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{1 \times 5}}$$

$$V_p = \underline{1.34 \times 10^8 \text{ m/s}}$$

$$\lambda = \frac{V_p}{f}$$

$$f = \frac{V_p}{\lambda} = \underline{26.83 \text{ MHz}}$$

---

(4) A 1MHz plane wave is propagating in fresh water where $\mu_r = 1$, $\varepsilon_r = 81$ & $\varepsilon_r'' = 0$. calculate (a) phase shift constant. (b) Wavelength (c) phase velocity (d) η (5) If amplitude of $\vec{E}$ field is 0.1V/m field E_y & H_z if wave is propagating in x-direction.

Soln:-

(a) $\beta = \omega \sqrt{\mu \varepsilon}$

$$\omega = 2\pi f = 2\pi \times 1 \times 10^6 = \underline{6.28} \text{ m}$$

$$\beta = 6.28 \times 10^6 \sqrt{8.854 \times 10^{-12} \times 81 \times 1 \times 4\pi \times 10^{-7}}$$

$$\beta = \underline{0.19 \text{ rad/m}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.19} = 33.31m$$

$$V_p = \frac{w}{\beta} = \frac{2\pi \times 10^6}{0.19} = 33.06 \times 10^6 \, m/sec$$

$$\eta = 377\sqrt{\frac{1}{81}} = \frac{377}{9} = 41.88\,\Omega$$

$$E_y = 0.1 \cos(wt - \beta x)$$

$$= 0.1 \cos(2\pi 10^6 t - 0.19x) \, V/m$$

$$H_z = \frac{E_y}{\eta} = 2.38 \times 10^{-3} \cos(2\pi 10^6 t - 0.19x) \, A/m$$

---

(5) A 9.375 GHz plane wave is propagating in Polyethylene $\varepsilon_r' = 2.26$, & σ. if amplitude of electric field intensity is 500 V/m & the material is assumed to be lossless find (a) β (b) λ (c) V_p (d) η (e) amplitude of $\vec{H}$

Soln:-

$$\beta = 2\pi \times 9.375 \times 10^9 \sqrt{\mu_0 \varepsilon_0 \varepsilon_r'}$$

(a) $$\beta = 2\pi \times 9.375 \times 10^9 \sqrt{4\pi \times 10^{-7} \times 8.854 \times 10^{-12} \times 2.26}$$

$$\beta = 295.37 \text{ rad/m}$$

(b) $$\lambda = \frac{2\pi}{\beta} = \frac{2 \times \pi}{295.37} = 0.01m$$

(c) $$V_p = \frac{w}{\beta} = \frac{2\pi \times 9.375 \times 10^9}{295.37} = 1.99 \times 10^8 \, m/sec$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon_1}} = 377\sqrt{\frac{\mu_r}{\varepsilon_r'}}$$

$$= 377\sqrt{\frac{1}{2.26}} = \underline{\underline{250.77\,\Omega}}$$

$$H = \frac{E}{\eta} = \frac{500}{250.77} = \underline{\underline{1.99\ \text{A/m}}}$$

---

(6) For Copper $\sigma = 58\,\text{M S/m}$ & teflon $\sigma = 30\,\text{n S/m}$ $\varepsilon = 2.1\,\varepsilon_0$. Verify that at 1 MHz Copper is a good Conductor & teflon is a good insulator.

<u>Soln</u>:-

good Conductor $\frac{\sigma}{\omega\varepsilon} \gg 1$

<u>Copper</u>,

$$\frac{\sigma}{\omega\varepsilon} = \frac{58 \times 10^{6}}{2\pi \times 1 \times 10^{6} \times 2.1 \times 8.854 \times 10^{-12}}$$

$$= 0.496 \times 10^{12} \gg 1$$

thus Cu is good Conductor

teflon $\Rightarrow$

$$\frac{\sigma}{\omega\varepsilon} = \frac{30 \times 10^{-9}}{2\pi \times 1 \times 10^{6} \times 2.1 \times 8.854 \times 10^{-12}}$$

$$= 2.57 \times 10^{-4} \ll 1$$

thus teflon is a good insulator.

(7) Consider a plane wave propagating in water, with a frequency of wave is 2.5 GHz and $\varepsilon_r' = 78$, $\varepsilon_r'' = 7$

find α, β, γ, V_p, η lossy medium.

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]^{1/2}$$

$$= \frac{2\pi \times 2.5\times10^9}{3\times10^8}\sqrt{\frac{78\times1}{2}}\left[\sqrt{1+\left(\frac{7}{78}\right)^2} - 1\right]^{1/2}$$

$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

$\frac{c}{\sqrt{\mu_r\varepsilon_r}}$

$$= 20.74 \text{ N/m}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]^{1/2}$$

$$= \frac{2\pi \times 2.5\times10^9}{3\times10^8}\sqrt{\frac{78}{2}}\left[\sqrt{1+\left(\frac{7}{78}\right)^2} + 1\right]^{1/2}$$

$$= 464 \text{ rad/m}$$

wave length $\lambda = \frac{2\pi}{\beta} = 1.4$ cm

$$V_p = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = f\lambda = 3.8\times10^7 \text{ m/sec}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon'}}\left[\frac{1}{\sqrt{1-j\left(\frac{\varepsilon''}{\varepsilon'}\right)}}\right] \Rightarrow 377\frac{1}{\sqrt{78}}\left[\frac{1}{\sqrt{1-j\left(\frac{7}{78}\right)}}\right]$$

$$= 43 + j1.9\,\Omega$$

$$\eta = 43\angle 2.6\,\Omega$$

(7) Consider an incident wave of freq 1 MHz at a sea water. find loss tangent, Skin depth, Wavelength, phase velocity & in sea water
$\sigma = 4$ S/m & $\varepsilon_r' = 81$.

Soln:-
loss tangent $= \frac{\sigma}{\omega\varepsilon'} = \frac{4}{2\pi \times 10^6 \times 81 \times 8.854 \times 10^{-12}}$

$= 887 \gg 1$ good Conductor

Skin depth $\delta = \frac{1}{\sqrt{f\mu\sigma\pi}} = \frac{1}{\sqrt{\pi \times 10^6 \times 4 \times 4\pi \times 10^{-7}}}$
$= 0.25\,m = 25\,cm$

Wavelength $\lambda = 2\pi\delta = 1.6\,m$

Phase velocity, $V_p = \omega\delta = 2\pi \times 10^6 \times 0.25$
$V_p = 1.6 \times 10^6$ m/sec

---

(8) Given a non-magnetic material having $\varepsilon_r' = 3.2$ & $\sigma = 1.5 \times 10^{-4}$ S/m. Find the Values at 3 MHz for (a) loss tangent (b) attenuation Constant (c) phase constant (d) Intrinsic impedance.

soln:-
(a) loss tangent $= \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon'} = \frac{1.5 \times 10^{-4}}{2\pi \times 3 \times 10^6 \times \varepsilon_0 \times 3.2}$

$= 0.28$

α, β, η for wave propagation in a good dielectric [additional notes]

where $\frac{\sigma}{\omega\varepsilon} \ll 1$ i.e, $\frac{\varepsilon''}{\varepsilon'} \ll 1$

W.K.T $\quad \gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

$$= \sqrt{(j\omega\mu)(j\omega\varepsilon)\left[1+\frac{\sigma}{j\omega\varepsilon}\right]}$$

$$= \sqrt{(j\omega)^2\mu\varepsilon\left[1+\frac{\sigma}{j\omega\varepsilon}\right]} = j\omega\sqrt{\mu\varepsilon}\sqrt{\left(1+\frac{\sigma}{j\omega\varepsilon}\right)\times\frac{j}{j}}$$

$$= j\omega\sqrt{\mu\varepsilon}\left[1-\frac{j\sigma}{\omega\varepsilon}\right]^{1/2}$$

Expanding using binomial series

$$[1+x]^n = 1 + nx + n(n-1)\frac{x^2}{2!} + \cdots$$

where $n = \frac{1}{2}$, $x = \frac{-j\sigma}{\omega\varepsilon}$

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left[1+\left(\frac{1}{2}\right)\left(-\frac{j\sigma}{\omega\varepsilon}\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{-j\sigma}{\omega\varepsilon}\right)^2\right]$$

$$\alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\left\{1 - \frac{j\sigma}{2\omega\varepsilon} + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}$$

$$\alpha + j\beta = j\omega\sqrt{\mu\varepsilon} + \cancel{\omega}\sqrt{\mu\varepsilon}\,\frac{\sigma}{2\cancel{\omega}\varepsilon} + j\omega\frac{\sqrt{\mu\varepsilon}}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2$$

Separating the real & imaginary terms

$$\alpha = \sqrt{\mu\varepsilon}\,\frac{\sigma}{2\varepsilon} \Rightarrow \sqrt{\frac{\mu}{\varepsilon}}\,\frac{\sigma}{2}.$$

$$\beta = \omega\sqrt{\mu\varepsilon} + \omega\frac{\sqrt{\mu\varepsilon}}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2 = \omega\sqrt{\mu\varepsilon}\left[1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]$$

Intrinsic impedance

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

$$= \sqrt{\frac{j\omega\mu}{j\omega\varepsilon\left[1+\frac{\sigma}{j\omega\varepsilon}\right]} \times \frac{j}{j}} = \sqrt{\frac{\mu}{\varepsilon}\left[\frac{1}{1-\frac{j\sigma}{\omega\varepsilon}}\right]}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left[1-\frac{j\sigma}{\omega\varepsilon}\right]^{-1/2}$$

$$n = -\frac{1}{2}, \quad x = \frac{-j\sigma}{\omega\varepsilon}$$

using binomial expansion.

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + \dots$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{j\sigma}{\omega\varepsilon}\right)\right]$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left[1 + \frac{j\sigma}{2\omega\varepsilon}\right] \ \Omega$$

α, β, η for a lossy medium [Refer to solve Problems]

$\frac{\sigma}{\omega\varepsilon} = \varepsilon''$, $\varepsilon = \varepsilon'$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1} = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]^{1/2} \ \text{N/m}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}}\sqrt{1+\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}} = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]^{1/2} \ \text{rad/m}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}\left[\frac{1}{1-j\frac{\sigma}{\omega\varepsilon}}\right]} = \sqrt{\frac{\mu}{\varepsilon'}}\left[\frac{1}{\sqrt{1-j\left(\frac{\varepsilon''}{\varepsilon'}\right)}}\right] \ \Omega$$

3) Given a non-magnetic material having $\varepsilon_r' = 3.2\ \varepsilon_0$ $\sigma = 1.5\times10^{-4}$ S/m. Find the values at 3 MHz for (a) loss tangent (b) attenuation constant (c) phase constant (d) Intrinsic impedance.

Soln:-

(a) loss tangent $= \dfrac{\varepsilon''}{\varepsilon'} = \dfrac{\sigma}{\omega\varepsilon'} = \dfrac{1.5\times10^{-4}}{2\pi\, 3\times10^{6}\times 8.854\times10^{-12}\times 3.2}$

$= 0.28$ $\qquad 0.1 < \dfrac{\varepsilon''}{\varepsilon'} < 1$

(b) $\alpha = \dfrac{\sigma}{2}\sqrt{\dfrac{\mu}{\varepsilon'}} = \dfrac{1.5\times10^{-4}}{2}\sqrt{\dfrac{4\pi\times10^{-7}}{\varepsilon_0\times 3.2}} = 0.0157$ Np/m

(c) $\beta = \omega\sqrt{\mu\varepsilon}\left[1+\dfrac{1}{8}\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2\right]$

$= 0.11$ rad/m

(d) $\eta = \sqrt{\dfrac{\mu}{\varepsilon'}}\left[1-\dfrac{3}{8}\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2 + j\dfrac{\sigma}{2\omega\varepsilon'}\right]$

$= 210.6\left[1+j0.140\right]$

$= 212.1\angle 7.9\ \Omega$

For Reference Only

The time average poynting vector (power density) is measured so, integrate S_x over one full cycle.

$$S_{avg} = \frac{1}{T}\int_0^T S_x \, dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}\frac{E_{ym}^2}{|\eta|} e^{-2\alpha x}\left[\cos(2\omega t - 2\beta x - \theta_n) + \cos\theta_n\right]dt$$

$$= \frac{E_{ym}^2}{2T|\eta|} e^{-2\alpha x}\left\{\int_0^T \cos(2\omega t - 2\beta x - \theta_n)\,dt + \int_0^T \cos\theta_n \, dt\right\}$$

$$= \frac{E_{ym}^2}{2T|\eta|} e^{-2\alpha x}\left\{0 + \cos\theta_n T\right\}$$

$$\boxed{S_{avg} = \frac{E_{ym}^2}{2|\eta|} e^{-2\alpha x}\cos\theta_n} \quad W/m^2$$

1) at frequencies 1 MHz, 100 MHz & 3000 MHz the dielectric constant of ice made from pure water has values of 4.15, 3.45 & 3.20 respectively while the loss tangent is 0.12, 0.035 & 0.0009 also respectively. If a uniform plane wave with an amplitude of 100 V/m at Z=0 is propagating through such ice, find the time average power density at Z=0 & Z=10m for each frequency.

Soln :-

$$S_{avg} = \frac{1}{2}\frac{E_x^2}{|\eta|} e^{-2\alpha z}\cos\theta_n$$

at $f = 1\,MHz$, $\varepsilon_r' = 4.15$ & $\frac{\sigma}{\omega\varepsilon'} = 0.12$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left(1 + \frac{j\sigma}{2\omega\varepsilon}\right) = \sqrt{\frac{4\pi\times10^{-7}}{8.854\times10^{-12}\times4.15}}\left(1+j\frac{0.12}{2}\right)$$

$$= 185.26\angle 3.43\ \Omega \Rightarrow |\eta| \angle\theta_n$$

1) Considers a plane wave propagating in water with frequency of water is 2.5 GHz & $\varepsilon_r' = 78$ & $\varepsilon_r'' = 7$. Calculate (a) α, (b) β, (c) λ (d) v_p (e) η

Soln:-

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\left(\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}-1\right)^{1/2}$$

Note $\left[c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\right]$

$$\alpha = \frac{2\pi\times 2.5\times 10^9}{3\times 10^8}\sqrt{\frac{78}{2}}\left[\sqrt{1+\left(\frac{7}{78}\right)^2}-1\right]^{1/2}$$

$W = 2\pi f$

$$= \underline{\underline{21}}\ \text{Np/m}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\left(\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}+1\right)^{1/2}$$

$$= \frac{2\pi\times 2.5\times 10^9}{3\times 10^8}\sqrt{\frac{78}{2}}\left(\sqrt{1+\left(\frac{7}{78}\right)^2}+1\right)^{1/2}$$

$$= \underline{\underline{464}}\ \text{rad/m}$$

$$\lambda = \frac{2\pi}{\beta} = \underline{\underline{1.4\,\text{cm}}}$$

$$v_p = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = f\lambda = 3.\underline{\underline{8}}\times 10^7\ \text{m/sec}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon'}}\left(\frac{1}{\sqrt{1-j\left(\frac{\varepsilon''}{\varepsilon'}\right)}}\right)$$

$$= \frac{377}{\sqrt{78}}\left(\frac{1}{\sqrt{1-j(7/78)}}\right) = \underline{\underline{4}}3+j1.9\,\Omega$$

$\sqrt{\frac{1}{r\angle\theta}} = \sqrt{\frac{1}{r}}\angle{-\theta/2}$

$$= \underline{\underline{43.04\angle 2.53^\circ\ \Omega}}$$

2) A marshy soil is characterised by 10^{-2} S/m, $\varepsilon_r = 15$ & $\mu_r = 1$ at frequencies 60Hz, 1MHz, 100MHz, 1GHz & indicate whether the soil may be considered as a Conductor / dielectric

given $\sigma = 10^{-2}$, $\varepsilon_r = 15$, $\mu_r = 1$

$$\frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{2\pi f\varepsilon} = \frac{10^{-2}}{2\pi f \times 8.854 \times 10^{-12} \times 15}$$

at $f = 60Hz$

$$\frac{\sigma}{\omega\varepsilon} = 199.67K \gg 1$$

good Conductor

at $f = 1MHz$

$$\frac{\sigma}{\omega\varepsilon} = 11.98 \gg 1$$ good Conductor

at $f = 100MHz$

$= 0.1198\ Hz \ll 1$ Good dielectric

at $f = 1GHz$ $= 1.198 \times 10^{-3} = 0$, perfect dielectric

(b) since $0.1 < \frac{\varepsilon''}{\varepsilon'} < 1$

$$\therefore \alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon'}} = \frac{1.5\times10^{-4}}{2}\sqrt{\frac{4\pi\times10^{-7}}{\varepsilon_0\times3.2}}$$

$$= \underline{\underline{0.0157}} \text{ N/m}$$

(c) phase Constant, $\beta = \omega\sqrt{\mu\varepsilon}\left[1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]$

$$= \underline{\underline{0.11 \text{ rad/m}}}$$

(d) $\eta = \sqrt{\frac{\mu}{\varepsilon'}}\left[1-\frac{3}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2 + j\frac{\sigma}{2\omega\varepsilon'}\right]$

$$= 210.6\left[1+j0.140\right]$$

$$= \underline{\underline{212.1\angle 7.9}}\ \Omega$$

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(19) A marshy (wet) soil is characterised by 10^{-2} S/m $\varepsilon_r = 15$ and $\mu_r = 1$ at frequencies 60 Hz, 1 MHz 100 MHz & 10 GHz indicates whether the soil may be considered as a conductor or a dielectric or neither.

Soln:- $\sigma = 10^{-12}$, $\varepsilon_r = 15$, $\mu_r = 1$

$$\frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{2\pi f\varepsilon} = \frac{10^{-2}}{2\pi\times f\, 8.854\times10^{-12}\times15}$$

$$\frac{\sigma}{\omega\varepsilon} = \frac{11.98\times10^{6}}{f}$$

at $f = 60$ Hz

$$\frac{\sigma}{\omega\varepsilon} = \frac{11.98\times10^{6}}{60} = 199.67\text{ kHz} \gg 1$$

= Good Conductor

(b) at $f = 1\,MHz$

$$\frac{\sigma}{\omega\varepsilon} = \frac{11.98 \times 10^6}{1 \times 10^6} = 11.98\,Hz > 1$$

$\therefore$ Good Conductor

(c) $f = 100\,MHz$

$$\frac{\sigma}{\omega\varepsilon} = \frac{11.98 \times 10^6}{100 \times 10^6} = 0.1198\,Hz$$

$\therefore$ Good dielectric

(d) $f = 1\,GHz$

$$\frac{\sigma}{\omega\varepsilon} = \frac{11.98 \times 10^6}{1 \times 10^9} = 1.198 \times 10^{-3} \Rightarrow 0$$

$\therefore$ perfect dielectric.

2) Consider a material for which $\mu_r = 1$, $\varepsilon_r' = 2.5$ & the loss tangent is 0.12. If these 3 values are constant with frequency in the range 0.5 MHz $\le f \le$ 100 MHz. Calculate (a) σ at 1 & 75 MHz (b) λ at 1 & 75 MHz (c) V_p at 1 & 75 MHz.

Soln:- loss tangent $= \frac{\sigma}{\omega \varepsilon'}$

(a) $\sigma = (\text{Loss tangent})(\omega \varepsilon')$

σ at 1 MHz $= (0.12)(2\pi)(10^6)(\varepsilon_0)\varepsilon_r'$

$= 1.66 \times 10^{-5}$ S/m

σ at 75 MHz $= 1.25 \times 10^{-3}$ S/m

(b) λ at 1 MHz $= \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon'}} = \frac{\cancel{2\pi}}{\cancel{2\pi} f\sqrt{\mu\varepsilon'}} = 189.7$ m

Ily λ at 75 MHz $= 2.529$ m

(c) $V_p = \frac{\omega}{\beta} = \frac{\cancel{2\pi f}}{\cancel{2\pi f}\sqrt{\mu\varepsilon'}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r'}}$

$= 3 \times 10^8 \times \frac{1}{\sqrt{2.5}} = 189 \times 10^6$ m/sec

V_p at 75 MHz $= 189 \times 10^6$ m/sec

$= 1.89 \times 10^8$ m/sec

(4)

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\omega\varepsilon' \times 0.12}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$\frac{\sigma}{\omega\varepsilon'} = 0.12$$
$$\sigma = 0.12\,\omega\varepsilon'$$

$$= \frac{2\times\pi\times 1\times10^{6}\times 8.854\times10^{-12}\times 4.15\times 0.12}{2}\sqrt{\frac{1\times4\pi\times10^{-7}}{8.854\times10^{-12}\times4.15}}$$

$$= \underline{\underline{2.5\times10^{-3}}}\ \text{Np/m}$$

$$S_{av}\Big|_{z=0} \Rightarrow \frac{1}{2}\frac{(100)^2}{185.26}e^{-2\times2.5\times10^{-3}\times0}\cdot\cos(3.43)$$

$$= \underline{\underline{26.99}}\ \text{W/m}^2$$

$$S_{avg}\Big|_{z=10m} = \frac{1}{2}\frac{(100)^2}{185.26}e^{-2\times2.5\times10^{-3}\times10}\cos(3.43)$$

$$= \underline{\underline{25.7}}\ \text{W/m}^2$$

at $f = 100\text{MHz}$, $\varepsilon_r' = 3.45$, $\frac{\sigma}{\omega\varepsilon'} = 0.035$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{0.035}{2}\right) = 202.83(1 + j0.0175)$$

$$= 20\underline{\underline{2.86}}\ \angle 1.003^\circ\ \Omega$$

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\omega\varepsilon\times0.035}{2}\sqrt{\frac{\mu}{\varepsilon}} = 0.\underline{\underline{06}}82$$

$$S_{avg}\big|_{z=0} = \frac{1}{2}\frac{(100)^2}{202.86}\cos(1.003) = 24.65\ \text{W/m}^2$$

$$S_{avg}\big|_{z=10m} \Rightarrow \frac{1}{2}\frac{(100)^2}{202.86}\cos(1.003)\,e^{-2\times0.0682\times10}$$
$$= 6.\underline{\underline{31}}\ \text{W/m}^2$$

at $f = 3000\text{MHz}$, $\varepsilon_r' = 3.20$, $\frac{\sigma}{\omega\varepsilon'} = 0.0009$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{0.0009}{2}\right) = 210.601(1 + j4.5\times10^{-4}) = \underline{\underline{210.6}}\angle 0.026$$

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\omega\varepsilon\times0.0009}{2}\sqrt{\frac{\mu}{\varepsilon}} = 0.\underline{05}06$$

$$S_{avg}\big|_{z=0},\ \frac{1}{2}\frac{(100)^2}{210.6}\cos(0.026) = \underline{\underline{23.74}}\ \text{W/m}^2$$

$$S_{avg}\big|_{z=10m} \Rightarrow \frac{1}{2}\frac{(100)^2}{}\cos(0.026)\,e^{-2\times0.0506\times10} = 8.63\ \text{W/m}^2$$